MATHEMATICAL PAPERS

BY

WILLIAM KINGDON CLIFFORD.

EDITED BY ROBERT TUCKER,

WITH AN INTRODUCTION BY H. J. STEPHEN SMITH.

"If he had lived we might have known something."

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MATHEMATICAL PAPERS

BY

My dear Mrs Clifford,

When towards the end of April 1879 I called upon you in answer to an urgent request to do so, it was with much surprise that I heard from you that it was one of your lamented Husband's last requests in regard to his work that I should be one of two persons to be asked to see the remaining Manuscript of his admirable book on *Dynamic* through the press. Though much gratified by his kind approval of what I had written upon the portion already in print, I could hardly look upon myself as qualified to undertake such a task—who indeed could hope to succeed in fitly editing fragments which had not received his last energizing touches? Whatever diffidence may have at first deterred me, yielded on further reflection to your pressing solicitation.

At this same interview you told me that it had been suggested, I believe by Mr Spottiswoode and Mr Macmillan, that a collected edition of Clifford's Mathematical Works, including the Memoirs already printed as well as such posthumous papers as might

be found in a state fit for publication, should be brought out under the auspices of a double editorship, one editor to be a resident at Cambridge and the other to be myself. If I hesitated to undertake the editing of the Manuscript of the *Dynamic*, you may believe I was even more afraid to undertake so great a responsibility as the editing of the Papers. Finding however at an interview I had with Dr Jack that it was desired that the papers should be brought out with all speed, and learning from Mr Spottiswoode that the plan of a double editorship had fallen through, I consented to do the best I could, feeling the less hesitation in making the attempt as I was able to obtain the assistance of Mr Spottiswoode himself and of Professors Cayley, Henrici and Smith.

Thus much of explanation I feel to be due from me to show that I did not engage in so great an undertaking of my own mere motion, but that I entered upon my task with no slight idea of its magnitude, although the actuality has far surpassed my expectation.

Your own assistance and kind interest in the progress of the work have been most valuable and most encouraging to me; and I can now only wish that the result of my efforts may deserve your approval, and may be not unworthy of the illustrious mathematician, your much loved Husband.

"If I have done well, it is that which I desired: but if slenderly and meanly, it is that which I could attain unto."

I need not enter into any account of the causes which have operated to bring about so tardy a completion of my labours. Sufficient it is for me to know that you have understood and allowed for the many hindrances which have arisen from circumstances connected with the fragmentary condition of some of the papers, and from the pressing claims upon my own time and upon that of the eminent mathematicians, whose advice and assistance have so ungrudgingly been extended to me.

Thanking you, my dear Mrs Clifford, for the honour you have conferred upon me in entrusting to me the work of raising this monument to the memory of one you love so well,

I remain,

Yours faithfully,

R. Tucker.

December 30th, 1881.

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CORRIGENDA.

On p. 22, *for* xv. *read* xvii.

,, 114, 6 up, *read* convertical.

,, 134, in connexion with the formulæ, refer to *Quarterly Journal of Mathematics*, No. 25. For this reference I am indebted to Mr J. J. Walker, who also suggests the following corrections "on p. 135, 4 up, *for* κ *read* $\frac{1}{4}$; p. 136, *omit* $\left[\frac{1}{\kappa}\right]$, l. 10, -2ι, l. 11."

,, 205, r is v in Clifford's MS. The paper is printed as in the *London Math. Society's Proceedings*.

,, 621, l. 12, *read* vol. xxxv. p. 21.

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PREFACE:

I. BIOGRAPHICAL.

II. BIBLIOGRAPHICAL.

BY

ROBERT TUCKER, M.A.,

HON. SEC. TO LONDON MATHEMATICAL SOCIETY,
SOMETIME SCHOLAR OF ST JOHN'S COLLEGE, CAMBRIDGE.

I.

WILLIAM KINGDON CLIFFORD,

Born at Exeter, May 4th, 1845. Died at Madeira, March 3rd, 1879.

AFTER the touching Biography prefixed to the *Lectures and Essays*, written with consummate literary skill by one who knew and loved Clifford well—"as his own soul,"—I feel it is unnecessary for me to say more than a few words as a preface to the present collection of his Mathematical Papers, and in the few following lines I shall confine myself to the simple statement of facts bearing upon his life and work as a Mathematician.

Clifford received his early education at Mr Templeton's school in Exeter, where in 1858 and 1859 he gained numerous distinctions in a very extended range of subjects at the University Local Examinations, both of Oxford and Cambridge. From this school he proceeded, in 1860, to King's College, London, where in like manner his efforts were crowned with success in more than one branch of study. Having in 1863 obtained a minor scholarship, he entered at Trinity College, Cambridge; at the end of his course in 1867 he graduated Second Wrangler and gained the Second Smith's Prize. In 1868, he was elected a Fellow of his College. He proceeded to M.A. in 1870; joined the Eclipse Expedition to Sicily of the same year, and was in 1871 elected Professor of Applied Mathematics and

Mechanics in University College, London, a post he occupied until the time of his death.

In June, 1874, Clifford was elected a Fellow of the Royal Society. He became a Member of the London Mathematical Society, June 18th, 1866, and served on its Council for every session from 1868—9 to 1876—7. Though he then ceased to be a member of the Council, he continued to take a warm interest in the proceedings of the Society, and on many a night he was present at the meetings when he was far from well and ought to have remained at home. I go back in imagination to the first meeting he attended—his was ever a welcome face in our room at University College and subsequently at Burlington House and Albemarle Street—it was on the evening of January 23rd, 1868, when Dr Hirst, our Treasurer, occupied the chair in the absence of our then President, Prof. Sylvester. Of the little band then present four have since been removed by death, viz., Clifford, Clerk Maxwell, Archibald Smith, and Thomas Cotterill. Clerk Maxwell made two communications: i. "on the construction of stereograms of surfaces," ii. "on the doctrine of Reciprocal Diagrams of Forces with the extension of Airy's Function of Stress from two dimensions to three." The former was made interesting by the fact that the author brought with him a real Image Stereoscope constructed after his own directions, and which invested the stereograms with a marvellous resemblance to the solids of which they were the plane presentments[1]. It is possible that Clifford spoke upon both these papers, but I have no record of his having done so. Mr J. J. Walker read a third paper in connection with which Clerk Maxwell asked him if he could point out a method of determining in what cases all the possible parts of the impossible roots of an equation are negative. In studying the motion of governors for regulating machines, he had found that the stability of the motion depended on this

[1] These diagrams may be seen in Vol. ix of the *Quarterly Journal of Pure and Applied Mathematics*. The paper is entitled "On the Cyclide."

condition, which is easily obtained for a cubic but is more difficult to find in the case of equations of higher degrees. Clifford at once said that we obtain the condition required by forming an equation whose roots are the sums of the roots of the original equation taken in pairs and determining the condition that the real roots of this equation should be negative. This may serve to give an idea of his readiness and helpfulness at our meetings. He rarely rose to speak of his own accord, but when a direct appeal was made to him he was ever ready to contribute a few pertinent remarks of an eminently suggestive kind, often showing in the course of the discussion that he had carefully considered the subject for himself. In this way he often threw out hints which enabled others to advance still further the researches which they had hitherto almost looked upon as their own. His readiness is exemplified in the following extract from a letter I have received from his private tutor, the Rev. Percival Frost[1]: "We were capital friends, yet I was so much engaged with a large number of pupils that I did not see very much of him except in a professional way. Even when he came to see me out of his working hours we used to get upon some mathematical curiosity, and both being fond of mathematics for their own sakes, we have often pursued our amusement into the small hours—once between 2 and 3—for which his tutor called him to account, good-naturedly excusing him when he heard of how he had been occupied. He often used to amuse me by solving in his head difficult problems, when some conversation like the following would take place. *Fr.* The men in the next room tell me this problem won't come out: there must be a mistake: just read it over and tell me where the setter has blundered. *Cl.* (reads it over and thinks a few minutes) I see how it is, he has, &c., &c."

Few anecdotes of Clifford's young days which show any foreshadowing of his future mathematical power have reached

[1] See also *Nature* (March 13th, 1879).

me. The following illustration, though trifling, may find a place here. His aunt (Mrs McLeod) writes: "In the year of the first Exhibition, 1851, his parents came to town, and I had the charge of him with my own child. When putting him to bed one night, I saw dear Willie looking very thoughtful. When I asked him what it was all about, he looked up in his smiling, loving way, saying, 'Aunt Annie, I don't think you would know.' But on my asking again, he said he was calculating 'how many edges (*sharp*) of a penknife it would take to go round the wheel of a coach.'

"We had a little talk about it, for it seemed impossible to me he could arrive at any conclusion. He then gave the figures and I daresay the size of the wheel: but all that I have long forgotten." The question and answer were submitted to an uncle, the late Mr Frank Kingdon, who was no mean mathematician, and he said the result was correct within a few figures.

A passage in Mr Pollock's Introduction to the *Lectures and Essays* (p. 41) would lead one to infer that Clifford turned his attention to kites at a somewhat late period of his life: it is however clear from a letter he himself wrote to Mr Miller that he had practised the flying of kites when a boy, for in 1863 he says "I have had in my mind, almost from the time I began to fly kites (I have not yet left off), the problem of finding the form of a kite-string under the action of the wind. On a rough trial, the other day, the intrinsic equation seemed not very difficult to obtain; if I get any result, I will send it to you hereafter[1]."

One more anecdote which will give an idea of the quick-

[1] The question figures as 6009 in the *Educational Times*, July 1879, May 1880 (Reprint, vol. xxxiii. p. 59). In the same letter he adds, "I have been trying to construct a second interpretation of mechanical equations similar to that of tangential co-ordinates, but have failed hitherto. Being a firm believer in the duality of symbols, I should look upon complete failure as a proof that our symbolical system is wrong." Cf. *Lectures and Essays*, p. 41, "his thoughts often ran upon mechanical inventions."

ness and clearness of his perception of complicated relations in space, must close this brief sketch. Mr Frost, in his previously cited letter, goes on to say, "My brother A. H. Frost, who was in England for a short holiday from his missionary work in India, brought with him a complicated puzzle which was to be taken to pieces. It was not a two-dimension one, like many, but solid: about as big as a good snowball. My brother said, 'I have heard you talk of the wonderful capacity of Clifford, prove it to me by asking him to tea, and I will believe you if he can take my puzzle to pieces.' I accordingly asked him, and on my brother's giving him the thing, he, without fingering it but simply looking it over for a few minutes, put his head in his hands for some ten minutes and then took hold of the puzzle, and at once, to my brother's astonishment, dislocated it, and my brother believed in him ever afterwards."

These few personal details of one who stood in the front rank of the mathematicians of our time, though trivial, will not, I think, be deemed out of place in such a volume as this. Ever kindly and unselfish, Clifford maintained in his relations with his brother mathematicians the same amiable bearing which endeared him to a large circle of private friends.

II.

If it were possible to ascertain the chronological order of creation of Clifford's papers, that order would have been followed in this work: but lacking the guidance of the 'vanished hand,' I soon found that I could not attempt such an arrangement and reconciled myself, unwillingly I must say to the publication of them in almost any order. In adopting this plan I received the approval of all the mathematicians whom I consulted upon the matter. In the following Bibliographical sketch, I have endeavoured to overcome the defects of this arrangement by assigning in all cases, to the best of my ability, the lower limit of publication or, in the case of posthumous papers, of composition. It is possible that some readers may be able to furnish data for correcting my statements: such information will be gratefully accepted.

When the number of a "Reprint" problem is given in black type, this shows that the *solution* is Clifford's, but not the *problem**. The titles of unpublished papers are printed in italics.

The MSS even the roughest, are all very neatly written and by the style of penmanship indicate four or five different epochs: it is by attention to this fact that I have been led to assign the respective dates to the posthumous papers.

* "Reprint" stands for "Mathematical Questions, with their Solutions, from the *Educational Times*," edited by W. J. C. Miller; "L. and E." stands for "Lectures and Essays," edited by Leslie Stephen and F. Pollock.

A lithographed specimen of the smallest handwriting faces the title-page*.

1863.

"On Jacobians and Polar Opposites," VI. pp. 23—33.

"Analogues of Pascal's Theorem," Nov. 20th, X. pp. 72—79.

Reprint, 1362, **1373**, 1378, **1379**, 1387, **1389**, **1393**, 1399, 1409, 4097 (1415)†, **1418**, 1423, 1448, 4143 (1459).

1864.

"Analytical Metrics" (see footnote, p. 80) XI. pp. 80—109.

"Geometrical Theorem," XLVII. pp. 410, 411.

Reprint: **1319**, **1394**, **1416**, **1421**, **1442**, **1443**, 1468, 1479, 1486, **1505**, 1507, **1514**, **1517**, **1519**, 2108 (1526), 1585, 1605.

1865.

"On Triangular Symmetry," XLVIII. pp. 412—414.

Reprint: 1497, 1638, 1652, 1675, **1679**, **1680**, 1691, 1724, **1732**, **1733**, 1748, 1750, 1775, 1795, 1823.

1866.

"On some extensions of the fundamental proposition in M. Chasles's Theory of Characteristics," March, XLIX. pp. 415—418.

"On the General Theory of Anharmonics," Nov. 22. XII. pp. 110—114.

"Bitangent Circles of a Conic," pp. 543—545, and

"Of Power-coordinates in General," pp. 546—555, may belong to this year.

Reprint: 1878, **1888**, 4199 (1907), 1918, 1929, 1962, 1996, 2220, 2229, 2253, 4696 (2281), 2301.

* Professor Henrici tells me that Clifford proposed to him that they should write in conjunction a series of books on Mathematics, beginning at the very commencement and carrying the subject in each case rapidly to the most advanced stages. The question was often discussed during the years in which Clifford was in full health, but this sheet is all that remains. Professor Henrici's occupation as Examiner to the London University prevented his actively commencing his part of the work. The short abstract xi. p. 650, is suggestive as to the place which some of the papers published in this volume were intended to take in a more extended scheme.

† The number in the bracket is that belonging to the first proposal of the question, the other number is that of the re-proposed question to which the solution was given.

On Question 2229, M. Chasles (Rapport sur les Progrès de la Géométrie, p. 353), writes:

"Ces surfaces anallagmatiques du quatrième ordre n'ont point tardé à fixer l'attention des géomètres. M. W. K. Clifford, notamment, en a fait connaitre plusieurs propriétés (voir *Ed. Times*, Sept. 1866, p. 134)."

1867.

"On the Principal axes of a Rigid body," VII. pp. 34—37.
Reprint: 2343, 2383, 2446, 2510, 2522.

1868.

"On some of the conditions of Mental Development," March 6. L. and E., Vol. I., pp. 75—108.
"On some Porismatic Problems," Nov. 9, III. pp. 17—19.
"On a general investigation of the theory of Polars," Nov. 26. XIII. pp. 115—118.
"On the Powers of Spheres" (?) *XXXIV. pp. 332—336. I assign the origin of this paper to this year, see my note, p. 332, though I am disposed to think that the paper was not actually written out long before 1876.
Reprint: 3885 (2674), 2732, 2748, 2776, 2793.

1869.

"On the Theory of Distances," XVI. pp. 130—164.
"On Syzygetic relations among the powers of Linear Quantics," Nov. 25. XIV. pp. 119—122.
"On Syzygetic relations connecting the powers of Linear Quantics." *XV. pp. 123—129.
"*On the Umbilici of Anallagmatic surfaces.*" Title only given in the British Association Report for this year.
"*Lectures on Geometry,*" given to a Class of Ladies at S. Kensington. Syllabus given, pp. 628—637.
"On Boundaries in General," printed at end of "Seeing and Thinking," see infra; also published in Macmillan's Magazine, August, 1879.
"Analysis of Cremona's Transformations" (?) pp. 538—542.
Reprint: 4236 (2817), 2858, **2923**, **2924**, **2932**, 2942, 2960, 2979, 5626 (3000), 3021.

1870.

"On a case of Evaporation in the order of a Resultant," Feb. 10, XVII. pp. 165—167.

"On Theories of the Physical Forces," Feb. 18. L. and E., Vol. I. pp. 100—123.

"Proof that every rational Equation has a root," Feb. 21. IV. p. 20.

"On the Space-theory of Matter," Feb. 21. V. pp. 21, 2.

"Synthetic Proof of Miquel's Theorem," March. VIII. pp. 38—54.

"*On an unexplained contradiction in Geometry.*" Title only in British Association Report of this year.

"Lecture Notes," pp. 524—530.

Reprint: 3197, 3255, 3282.

1871.

"*The History of the Sun: being an explanation of the nebular hypothesis and of recent controversies in regard to the time which can be allowed for the evolution of life.*" April 16. L. and E. Vol. I. p. 68.

"On a Canonical form of Spherical Harmonics." August. XXV. pp. 234, 5.*

"*Note on the Secular Cooling and the Figure of the Earth.*" Title only in British Association Report for this year.

Reprint, 4034 (3308).

1872.

"Atoms." Jan. 7. L. and E., Vol. I. pp. 158—190.

"*Ether; the evidence for its Existence and the Phenomena it explains.*" April 14. L. and E., Vol. I. p. 68.

"Remarks on a Theory of the Exponential Function derived from the equation $\frac{du}{dt} = pu$." May 9. XLV. p. 406.

"*On Babbage's Calculating Machines.*" May 24. L. and E., Vol. I. p. 69.

"On the Aims and Instruments of Scientific Thought." August. L. and E., Vol. I. pp. 124—157.

* In *Nature*, Sept. 7, there is little more than the title of this paper given.

"*On the Contact of Surfaces of the Second Order with other Surfaces.*" Title only in British Association Report of this year.

An *Athenæum* correspondent says (Aug. 24), "Prof. Clifford explained that the radial polarization settles the fact of there being floating clouds of solid or liquid matter in the corona." No. 2339.

"On a Theorem relating to Polyhedra, analogous to Mr Cotterill's Theorem on Plane Polygons." Nov. 14. XVIII. pp. 168—176.

"*The Dawn of the Sciences in Europe.*" Nov. 17. L. and E., Vol. I. p. 68.

"Geometry on an Ellipsoid." Dec. 12. XIX. pp. 177—180.

1873.

"The Philosophy of the Pure Sciences." March 1, 8, 15. L. and E., Vol. I. pp. 254—340.

"On the Hypotheses which lie at the bases of Geometry." May 1, 8. IX. pp. 55—71.

"*The Relations between Science and some Modern Poetry.*" May 4. Recast as "Cosmic Emotion." L. and E., Vol. I. p. 68.

"Preliminary Sketch of Biquaternions." June 12. XX. pp. 181—200.

"On Mr Spottiswoode's Contact Problems." June 19. XXXII. pp. 287—304.

"*On some Curves of the Fifth Class,*" and "*On a Surface of Zero Curvature and Finite Extent.*" Titles only in British Association Report for this year.

"Review of De Morgan's 'Budget of Paradoxes.'" August 15. pp. 559—561.

"Graphic representation of the Harmonic Components of a Periodic Motion." Dec. 11. XXI. pp. 201—204.

"Syllabus of Lectures on Motion." (?) pp. 516—523.

Reprint: **3876**, 3961, 3980, 4010, 4069.

1874.

"Review of Vol. I. of G. H. Lewes' 'Problems of Life and Mind.'" Feb. 7. L. and E., Vol. I. p. 68.

"The First and Last Catastrophe." April 12. L. and E., Vol. I. pp. 191—227.

"*On the Education of the People.*" May 22. L. and E., Vol. I. p. 68.

"*On a Message from Prof. Sylvester,*" and "*On the General Equations of Chemical Decomposition.*" Titles only given in the British Association Report for this year.

"Prof. Clifford exhibited a jointed frame in illustration of some beautiful discoveries recently made in connexion with what machinists call parallel motion." *Athenæum*, Sept. 5. No. 2445 (alluding to the discoveries of Peaucellier, Hart, and Sylvester).

"This paper (*Chemical Decomposition*) was read before Section A. The Author thinks that Chemical Equations may be brought under a general formula. Thus, $H_2+Cl_2=2HCl$. If we assume that there is a structure common to the hydrogen and the chlorine atoms, also a structure confined to the hydrogen, and likewise a structure confined to the chlorine atoms, we may represent the equation thus: $XYY+XZZ=2\ XYZ$, when X represents the common structure, and Y and Z the structures which are confined to hydrogen and chlorine respectively. So $2H_2+O_2=2H_2O$ may be represented thus: $2\,XY+XXZZ=2\,XXYZ$. These equations involve no hypotheses, because the fundamental facts of the molecular theory are now firmly established. Reasoning from these and similar equations, the author deduces the result that the ordinary equations of chemistry, such as those just stated, are expressive of facts, and that the hydrogen molecule really consists of two equal atoms." *Nature*, Sept. 24, No. 256.

"Body and Mind." Nov. 1. L. and E., Vol. II. pp. 31—70.

"On the Nature of Things in themselves." L. and E., Vol. II. pp. 71—88.

"Seeing and Thinking." December. Published in 1879 (*Nature* Series).

"Motion of a Solid in Elliptic Space." *XLI. pp. 378—384. (See note, p. 378, but I do not think the paper was *written* until 1876).

1875.

"*The general features of the History of Science.*" Feb. 27, March 6, 13, 20. L. and E., Vol. I. p. 69.

"*Ultramontanism.*" April 28. L. and E., Vol. I. p. 68.

"The Unseen Universe." June. L. and E., Vol. I. pp. 228—253.

"On the Scientific Basis of Morals." L. and E., Vol. II. pp. 106—123.

"*On the Theory of Linear Transformations:* (i) *the Graphical representation of Invariants;* (ii) *the Expansion of Unsymmetrical*

Functions in Symmetrical Functions and Determinants; (iii) *the Notation of Matrices.*" Title only in British Association Report of this year.

"Prof. Clifford astonished the section by some remarkable applications of Grassmann's "polar multiplication" defined by the law that *ba* is minus *ab*. He applied it to the graphical representation of invariants, to the expansion of unsymmetrical functions, and to the notation of matrices, illustrating his remarks by drawings representing atoms hung together in various ways."—*Athenæum.* Sept. 11th. No. 2498.

"A Fragment on Matrices," *xxxv. pp. 337—341, perhaps belongs to this period.

"Right and Wrong: the scientific ground of their distinction." Nov. 7th. L. and E., Vol. II. pp. 124—176.

"On the Transformation of Elliptic Functions." Dec. 9th. xxii. pp. 205—217.

Reprint: 4641, 4819, 4843.

1876.

"On the Free Motion under no forces of a Rigid System in an *n*-fold homaloid." (Provisional Notice.) Jan. 13th. xxvi. pp. 236—240.

"*Sight and what it tells us.*" Feb. 24. L. and E., Vol. I. p. 69.

"Instruments used in Measurement." L. pp. 419—423; L. and E., Vol. II. p. 3—8.

"Instruments illustrating Kinematics, Statics and Dynamics." LI. pp. 424—440; L. and E., Vol. II. pp. 9—30.

"The Ethics of Belief." L. and E., Vol. II. pp. 177—211.

"Notes on the communication entitled 'On the Transformation of Elliptic Functions.'" xxii. pp. 218—228.

"On the Classification of Geometric Algebras," *xliii. pp. 397—401.

"On In-and-circumscribed Polyhedra" (?) *xxiv. pp. 229—233.

"On the Theory of Screws in a space of constant positive curvature" (?) *xliv. pp. 402—405.

"On Tricircular Sextics" (?) *xxxvi. pp. 342—345.

"On the Double Theta-functions" (?) *xl. pp. 368—377.

"Further Note on Biquaternions" (?) *xlii. pp. 385—396.

Reprint: 4871, 4897, 4925, 4950, 4972, 4996.

1877.

"On the Types of Compound Statement involving four classes." Jan. 9th: I. pp. 1—13. L. and E., Vol. II. pp. 89—106.

"The Ethics of Religion." March 4. L. and E., Vol. II. pp. 212—243.

"The Influence upon Morality of a Decline in Religious Belief." April: L. and E., Vol. II. pp. 244—252.

Review of Dr Booth's "New Geometrical Methods." June: pp. 562—564.

"On the Canonical Form and Dissection of a Riemann's Surface." June 14th: XXVII. pp. 241—254.

"Cosmic Emotion." October: L. and E., Vol. II. pp. 253—285.

"Notes on Vortex-Motion, on the Triple-generation of Three-bar Curves, and on the Mass-centre of an Octahedron." November 8th: XLVI. pp. 407—409.

"Enumeration of the Types of Compound Statements." (?) * II. pp. 14—16.

In the Michaelmas Term of this year (cf. L. and E., Vol. I. p. 27) Prof. Clifford delivered a course of ten lectures on Quaternions with a view to their physical applications, for students of Physics, "who are not able or willing to read very high Mathematics."

For an admirable collection of Notes of these Lectures (see *infra*, pp. 478—515), I am indebted to the kindness of Miss Ellen Watson*, who also placed at my disposal other Lecture Notes which I have not made use of in the present work. Mr G. Griffith, of Harrow, who also attended the Course, lent me his skeleton Notes which enabled me to clear up one or two minor points.

Two other courses were announced. In the Lent Term of 1878, "On Elliptic Functions and some of their Physical Applications treated on a basis of Elementary Algebra, but assuming a Knowledge of the Elements of the Differential Calculus†."

* Miss Watson was the first woman to enter the Classes of Mathematics at University College, London. Of her, Clifford said, "that her proficiency would have been very rare in a man," and that "he was totally unprepared to find it in a woman."—She was obliged to leave England on account of failing health, and died in December 1880, at Grahamstown, South Africa.

† One or two Lectures of this course were delivered, but the few Notes of them in Miss Watson's MSS. are too fragmentary to be of service.

These were to be followed by a third Course in the Midsummer Term, "On Spherical Harmonics and other functions rising out of the Theory of the Potential and Allied Theories treated by means of Partial Differential Equations and series with special attention to problems of Electricity."

"Algebraic Introduction to Elliptic Functions" (?commenced and added to at different times), pp. 443—477.

"On Groups of Periodic Functions" (?) * XXXVIII. pp. 350—355. Reprint: 5304, 5457.

1878.

"Remarks on the Chemico-Algebraical Theory." XXVIII. pp. 255—257.

"Virchow on the Teaching of Science." April: L. and E., Vol. II. pp. 286—321.

"On the Classification of Loci." April 8: XXXIII. pp. 305—329.

"Childhood and Ignorance." May: L. and E., Vol. I. p. 70.

"Applications of Grassmann's Extensive Algebra." XXX. pp. 266—276.

"Elements of Dynamic: an Introduction to the Study of Motion and Rest in Solid and Fluid Bodies." Part I. Kinematic.

It may not be out of place to give here an analysis of what Clifford intended to give in Vol. II. of his book, the manuscript of which is now in my possession.

BOOK IV. FORCES.

Cap. I. The Laws of Motion.

Cap. II. The Conditions of Equilibrium of a Rigid Body.

Cap. III. The Composition of Forces.

§ 1. The Link-polygon, Reciprocal diagrams, &c.

§ 2. Centres of Inertia, Second Moment, Cores.

§ 3. Attractions, Potential and Level Surfaces, Sources and Sinks. Theorems of Stokes and Chasles. Electric Images. Centrobaric Bodies.

Cap. IV. Motion of a Rigid Body.

BOOK V. STRESSES.

Cap. I. Solids.

"Notes on Quantics of Alternate Numbers, used as a means for determining the Invariants and Covariants of Quantics in general." (?) *XXIX. pp. 258—265.

"Binary Forms of Alternate Variables." (?) *XXXI. pp. 277—286.

"On Bessel's Functions." *XXXVII. pp. 346—349.

"Theory of Marks of Multiple Theta-functions." (?) *XXXIX. pp. 356—367.

The manuscript of "The common sense of the Exact Sciences" is in Prof. Rowe's hands, and is almost ready for publication.

All introduced matter is, with one or two exceptions, included in [ ]: this has necessitated the adoption of another form of *bracket* when [ ] occurs in Clifford's papers. I am in the main responsible for these additions, though I have in all cases submitted the suggestions to one or more of the mathematicians mentioned in my dedicatory letter: their own additions are suitably initialled.

INTRODUCTION

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INTRODUCTION.

It will be generally admitted that the publication in a collected form of the works of the eminent men, who have moulded the mathematical sciences into their present form, has become little less than a necessity to those who desire to follow in their footsteps, and to advance, if possible, beyond the limits attained by them. And mathematicians will gratefully acknowledge that no inconsiderable progress has already been made towards satisfying this requirement. To the Academy of Sciences at Paris we are indebted for magnificent editions of the complete works of Laplace and of Lagrange; the Government of Norway has given to the world the collected memoirs of Abel; the Academy of Goettingen has fulfilled the same duty toward the great names of Gauss and Riemann; and the Academy of Berlin has followed the example by undertaking editions of the works of Steiner and Jacobi. In our own country we have the collected works of Green, of Mac Cullagh, of Gregory, of Leslie Ellis, and of Macquorn Rankine; not to mention the volumes of reprinted memoirs which we owe to living writers; for example, to Sir William Thomson and to Professor Stokes. Such collections, we may hope, will form an increasing portion of every scientific library. At the present time the results of mathematical research almost always appear in the Transactions of Societies, or in periodicals specially devoted to mathematical writings: the contents even of the most original treatises being generally anticipated by their authors in memoirs, which are often not wholly superseded

by the works themselves. But the number of the periodical repositories of mathematical literature has become so great, that papers consigned to them, although preserved, as we may hope, for all time, are in imminent danger of passing out of sight within a few years after their first appearance. They are preserved from destruction, but not from oblivion; they share the fate of manuscripts hidden in the archives of some great library from which it is in itself a work of research to disinter them. This "mislaying," if it may be so termed, of important memoirs is not only a loss to the history of science, but interferes seriously with the discovery of new knowledge. For notwithstanding the ardour with which mathematical investigation is at present pursued in every direction, a much longer time than is perhaps sometimes supposed elapses before a mathematical work of genuine originality, be it a brief note, or an elaborate treatise, becomes antiquated. It would be out of place in this connexion to mention the Principia of Newton, which stands apart by itself, and of which not only the methods and results, but even the very words have become the common property of all men of science. Nor need we even refer to works which have marked the beginning of a new epoch in their respective departments, such as the Mécanique Céleste, the Mécanique Analytique, the Disquisitiones Arithmeticæ, the Traité Analytique de la Chaleur, the Fundamenta Nova, or the Systematische Entwickelung der Geometrischen Gestalten—the freshness of which time has hardly impaired, while the superstructures which have been based upon them have added incalculably to their importance. But leaving out of count these and other great classics of the science, the trains of thought hidden in the opuscula of Euler, of Lagrange, of Gauss, of Poisson, of Cauchy, of Abel, and Jacobi, are still unexhausted; and, far from having lost their value by the lapse of time, have in many cases acquired an increased suggestiveness from the light which the more extended knowledge of recent times has thrown upon them.

Such a prospect of future and long-continued usefulness, we may venture to hope, awaits many of Clifford's memoirs; and, even more than the immediate interest attaching to them, justifies, if any justification be needed, their appearance in their present collected form.

It might be interesting to enquire why it is that mathematical writings retain a scientific (as opposed to a merely historical) value for a longer time than memoirs recording researches in the sciences of experiment and observation. Among many partial answers which might be given to this question, one is suggested by the character of many of Clifford's papers, and has its foundation in the nature of the subjects with which they deal. Speculation in pure mathematics resembles metaphysical speculation in this, that the whole universe of thought to which it refers is so closely inter-dependent, that a clear-sighted and powerful thinker cannot fix his mental vision (however keen his effort after concentration may be) on any one region in it, without catching glimpses of something that lies beyond, and without discerning, more or less dimly, new relations to be examined, and new lines of research, which may perhaps have no immediate relevancy to the particular enquiry in which he is engaged. And these glimpses, if recorded, or even if only half unconsciously indicated, in the account which he afterwards gives of his work, are not unlikely to suggest a wholly new departure to some kindred spirit in a future time. On this ground, more strongly perhaps than on any other, we may venture to commend the present volume to the rising generation of English mathematicians. The collection includes papers—some of them youthful efforts—some suggested, one might say casually, by the researches of scientific friends—which relate to special problems, and which nevertheless would be sufficient of themselves to establish a considerable mathematical reputation. There are others, again, the work of a maturer time, and planned with a wider scope, which are models of artistic perfection, in respect both of

the clearness and depth of the thought, and of the manner of its presentation. Lastly, besides these finished pieces there are others, rough-hewn and imperfect in execution, but conveying a still stronger impression of the fertility of invention, and of the far-reaching power of mental vision, with which Clifford was endowed. Some of these fragmentary records are full of great ideas, shadowed forth in outlines, not always free from indistinctness, but always suggesting long vistas of future discovery, the path to which seems for the moment to lie clear before his mind. Their very incompleteness reminds us how much the world has lost by losing him; and brings home to us the melancholy feeling that, however highly we may estimate the work which he actually accomplished during his brief life-time, he must nevertheless be counted among "the inheritors of unfulfilled renown."

But if the republication of Clifford's papers stands in no need of any justification, some apology is wanted for an Introduction which can offer but little interest to those who do not intend to study the volume itself, and which to those who do, must seem at the best superfluous. Perhaps, however, even in these days of increasing specialisation, there may still be found an intermediate class of readers, who are not mathematicians by profession, and who nevertheless do not regard analysis and geometry as volumes sealed except to the initiated few, but as belonging, in their results at least, to the whole world of science. Some persons, one is willing to believe, partly from a recollection of their own early studies, and partly from a general sympathy with all branches of intellectual activity, are disposed to follow with an appreciative, or at least an indulgent curiosity, the exposition of new mathematical ideas. For such friendly readers these pages are intended; and it would be strange if the class of persons, among whom they are to be found, has not been considerably increased by the admirable lectures[1] in which Clifford

[1] *Lectures and Essays*, by W. K. Clifford, Vol. I. pp. 254—340.

has himself analysed, in popular phraseology but with the utmost scientific precision, the fundamental principles of geometry and arithmetic, as they appear in the "fierce light" which has been turned upon them by a controversy in which both metaphysicians and mathematicians have taken part.

All then that can with any propriety be attempted here is, in the first place, to characterise some one or more of the principal trains of thought which seem to have exercised an abiding influence on Clifford's mind; and then to classify his memoirs in a few main groups, pointing out the central ideas of each group, and shewing, as far as possible, the interdependence of these ideas upon one another.

Clifford was above all and before all a geometer. Nine-tenths, and more, of the contents of this volume, including nearly all the systematic researches recorded in it, are geometrical. It is true that in the treatment of geometrical questions he shows a marked preference for symbolical methods; and, as might be expected, a marvellous command over analytical expression. It may even be true that the limitations involved in a scrupulous adherence to the methods of pure geometry would have been distasteful to him. Of his skill in the use of these special methods the "Problems and Solutions" so liberally contributed by him to the *Educational Times* afford abundant proof. But among his more elaborate papers there is perhaps but one, the "Geometry on an Ellipsoid," which would satisfy purists of the school of Poncelet and Chasles, as being wholly free from the contamination of analytical methods; and even in this beautiful application of the method of stereographic projection—in the generalized sense in which that method is used in modern pure geometry—the "imaginary circle at infinity" occurs in the first sentence. But, whatever the method employed—and in variety of method Clifford takes an evident pleasure—the properties of space remain the perpetual subject of his contemplation. Purely analytical researches undertaken without any impulse from or

reference to geometry, are few and far between. For even the Elliptic and Abelian functions were approached by Clifford from the side of geometry. His early note "On some Porismatic Problems," relating to the theorem of Poncelet, which asserts that "given two conics, a polygon of a given number of sides, all whose vertices shall lie on one of them, and all whose sides shall touch the other, can either not be drawn at all, or else can be drawn in an infinite number of different ways," led him to the study of the connexion, established by Professor Cayley, between this theorem and the addition of elliptic functions of the first species; and thus to the discovery of a geometrical theory of the transformation of elliptic functions, which forms the subject of one of his most brilliant investigations (XXII.). But in his further prosecution of the subject the elliptic functions again disappear from view, and he returns to the geometrical doctrine of correspondences, and to the theory of the polyhedra, of which the faces osculate a skew cubic curve.

In like manner it would appear that he was attracted to the consideration of the Abelian Integrals by their relation to another and widely different part of geometry, the Geometry of Situation, as it has been termed. His memoir on the "Canonical Dissection of a Riemann Surface," founded on the researches of Clebsch and Lüroth, contains the simplest account which has yet been given of this important chapter of a great theory; and the reduction of a Riemann surface to the surface of a solid having a certain number of holes through it, presents to the mind what is perhaps the clearest image which it is possible to obtain of the space of two dimensions upon which a many-valued algebraical function can be mapped with the same distinctness with which a one-valued function can be mapped on a plane. But the study of the Abelian Integrals—however geometrical may have been the form in which he first envisaged this theory, led him by an inevitable sequence of ideas to the Theta functions, which form the indispensable

basis for a study of the relations subsisting between the upper limits of a system of algebraical integrals, and the values of the integrals themselves. His posthumous and unfinished memoirs on the Double and Multiple Theta functions (XXXVIII., XXXIX. and XL.) form a not unimportant contribution to pure analysis; and, though now published long after their proper date, are, it may be hoped, neither too late, nor too incomplete, to exercise some influence on the development of this rapidly growing theory. The "Algebraical Introduction to Elliptic functions," which is in fact a treatise on the single Theta functions, probably took its rise in connexion with these ulterior researches; Clifford desiring to obtain a complete command over the manipulation of these series in the simplest cases, before proceeding to apply them to the general problem of the inversion of algebraical integrals.

Enough has been said to show that Clifford's predilection for geometry lay deep. But to this his favourite science he attributed the widest imaginable scope, and at times regarded it as co-extensive with the whole domain of nature. He was a metaphysician (though he would only have accepted the name subject to an interpretation) as well as a mathematician; and geometry was to him an important factor in the problem of "solving the universe[1]". Thus he was a geometer of a type peculiarly his own; and his dealings with the science were characterized by an amount of scepticism and an amount of faith which one would hardly expect to find combined in a mathematician. He had early read and translated Riemann's celebrated discourse (IX.) "On the Hypotheses which lie at the Basis of Geometry," and had imbibed the views set forth in it as a part of his intellectual nature. Some men who have an ardent love for new knowledge find it

[1] "At the Savile I met C.... and solved the universe with great delight until A. came in and wanted to take him off... Of course I would not let him go... We walked about in the New Road solving more universe." *Lectures and Essays*, Introduction, p. 30.

difficult to maintain an unflagging interest in geometry, because they regard it as a purely deductive science, of which the first principles (axioms, postulates and definitions), whether derived from experience or not, are unquestionable, and contain implicitly in themselves all possible propositions concerning space. Thus the unknown, or at least the unforeseen, seems to be excluded from geometry, because whatever may be found out hereafter must be latent in what is already known. But upon the view put forward by Riemann and adopted by Clifford, the essential properties of space have to be regarded as things still unknown, which we may one day hope to find out by closer observation and more patient reflection, and not as axioms to be accepted on the authority of universal experience, or of the inner consciousness.

These speculations had so much influence on a great part of Clifford's work that it may not be out of place to pursue the subject a little further. In his lecture "On the Postulates of the Science of Space[1]", he has stated his own views on the question with singular clearness and brilliancy; and the pages in which he has expressed them are likely to be remembered, as marking an important moment in the controversy concerning the nature of space and the origin of our knowledge of it, which is likely to last as long as metaphysical enquiries have any interest for mankind. In this lecture he enumerates four fundamental postulates on which the ordinary conception of space is founded, (1) its continuity, (2) its flatness in its smallest parts, (3) its similarity to itself at every point, or, which is the same thing, the possibility of the existence of the same figure in any two different places, (4) the possibility of the existence of figures similar to one another, but on different scales of magnitude. The second of these postulates requires some comment to make it intelligible. Perhaps the simplest account that can be given of a space which is flat in its smallest parts is, that if anywhere

[1] *Lectures and Essays*, Vol. I. pp. 295—325.

in it we take three points very near to one another and join them by the shortest lines that can possibly be drawn, the triangular figure so formed will lie very nearly in a plane; the mathematical equivalent of this statement being that the square of the distance between any given point and any other infinitely near to it can always be expressed as a homogeneous quadratic form, in which the indeterminates are the infinitesimal differences between the co-ordinates of the two points, and the coefficients are functions of the co-ordinates of the given point. If we go further and join one of the three infinitely near points by the shortest lines possible to every point on the line already joining the other two, the assemblage of these lines will form a triangular surface-element: if this surface-element is absolutely plane, whatever be the three infinitely near points which we have taken, the space is flat; if the surface-element has a finite curvature, the space, while retaining the property of elementary flatness, is said to have curvature; and this curvature is measured, for the surface direction determined by the three points, by the curvature of the surface-element which we have constructed.

As to the first postulate, Clifford indicates his readiness to adopt either of the two opposite hypotheses that space is continuous or that it is discontinuous, while admitting fully that no phenomena have yet been observed which point to its discontinuity.

Of the second postulate, in this respect following Riemann, he speaks in the same general terms; we must not shrink from rejecting it, if its rejection should be found to assist us in the explanation of natural phenomena. The postulate is not inconsistent with a hypothesis which at one time was a great favourite with him, and which he has described in a remarkable communication (v.), presented to the Cambridge Philosophical Society, in 1870. In this brief note, comprised within a single page, he appears to adopt the hypothesis (for his language on the point is not quite free from ambiguity), that space

has everywhere a finite curvature, but that this curvature is continually changing, and that all the phenomena of the universe may possibly consist in changes of the curvature of space. A finite curvature, it will be remembered, is consistent with, and indeed implies elementary flatness. Unfortunately Clifford, though in earlier days he was fond of discussing this theory, no doubt as one possible mode of "solving the universe," has left no memoranda relating to it, perhaps because the efforts which he made to work it out in detail, led him to no satisfactory conclusion. In the note of 1870 he speaks of it with a confidence which must not be taken too literally. He would probably have allowed that Lord Bacon's criticism on Gilbert, "postquam in contemplationibus magnetis se laboriosissime exercuisset, confinxit statim philosophiam consentaneam rei apud ipsum præpollenti," admitted of an application, *mutatis mutandis*, to his own effort to resolve all philosophy into geometry; though he would no doubt have maintained with the utmost depth of conviction that, for aught we know to the contrary, the properties of space may change with time.

But whatever importance he may have temporarily attached to the opinion that space may not be independent of time, this idea has left no other perceptible traces in his mathematical writings. Very different is the case with another hypothesis as to the nature of space, which is somewhat less widely divergent from ordinary conceptions, and to which Clifford appears at all times to have turned with peculiar favour.

This hypothesis admits the first three of the postulates enumerated above, as expressing true properties of space, but rejects the fourth, substituting for it the new postulate that space has a finite but very small curvature, which is approximately the same for any two points, and for any two surface directions at the same point. Admitting this postulate, we find ourselves in the presence of two alternatives, between which we have to choose. For we may imagine either that

the curvature of all surface-elements, constructed in the manner above described, has the same positive value, or that it has the same negative value; understanding by a positive curvature a curvature such as that of the outer portion of the surface of an anchor-ring, where the tangent plane at any point just meets the surface and does not cut it; and by a negative curvature a curvature such as that of the inner portion of the same surface, where the tangent plane cuts the surface at the point of contact. To the hypothesis that space has a constant negative curvature considerable historical interest attaches. For this hypothesis was first arrived at, not by following out such general views as those indicated by Riemann, but in a much more elementary manner. The celebrated twelfth axiom, as is well known, is the basis of Euclid's theory of parallel lines; and the assertion made in it is in fact equivalent to an assumption of the fundamental proposition of plane geometry, that the three angles of a triangle are equal to two right angles. It is now universally allowed that all efforts to demonstrate Euclid's axiom have failed; but the Russian mathematician, Lobatchewsky, appears to have been the first person to whom the idea occurred of dispensing with the axiom altogether, and trying to see what would become of geometry without it. The idea was obvious, but it was also profound; and Lobatchewsky was rewarded by the discovery that it is possible to construct a consistent and complete system of geometry upon the hypothesis that the three angles of a triangle are less than two right angles. Till the discovery of Lobatchewsky, the only substantial addition that had been made to Euclid's theory of parallel lines was a demonstration by Legendre, that the angles of a triangle cannot be greater than two right angles. As a matter of fact the demonstration of Legendre depends on the assumption that space is infinite; an assumption which, from the point of view taken by Riemann, cannot be regarded as justified by experience: but the considerations upon which the demonstration rests decided Lobat-

chewsky, as between the two alternative hypotheses that the angles of a triangle are less, and that they are greater, than two right angles, to adopt the former. This was in effect to adopt the hypothesis (though it does not appear to have occurred to Lobatchewsky in that light) that a plane has negative curvature. It was reserved for an Italian mathematician, Beltrami, to show that the plane geometry of Lobatchewsky is identical with the geometry of a pseudo-spherical surface, *i.e.* of a surface of constant negative curvature.

What Clifford thought of the philosophical importance of the work of Lobatchewsky the following quotation may serve to show:

"Each of them [Copernicus and Lobatchewsky] has brought about a revolution in scientific ideas so great that it can only be compared with that wrought by the other. And the reason of the transcendent importance of these two changes is that they are changes in the conception of the Cosmos. Before the time of Copernicus men knew all about the universe. They would tell you in the schools, pat off by heart, all that it was, and what it had been, and what it would be.... In any case the universe was a known thing. Now the enormous effect of the Copernican system, and of the astronomical discoveries that have followed it, is that, in place of this knowledge of a little, which was called knowledge of the universe, of Eternity and Immensity, we have now got knowledge of a great deal more; but we only call it the knowledge of Here and Now.... This then was the change effected by Copernicus in the idea of the universe. But there was left another to be made. For the laws of space and motion implied an infinite space and an infinite duration, about whose properties as space and time everything was accurately known. The very constitution of those parts of it which are at an infinite distance from us, 'geometry upon the plane at infinity,' is just as well known, if the Euclidean assumptions are true, as the geometry of any portion of this room.... So that here we have real knowledge of something at least that concerns

the Cosmos; something that is true of the Immensities and the Eternities. That something Lobatchewsky and his successors have taken away. The geometer of to-day knows nothing about the nature of actually existing space at an infinite distance; he knows nothing about the properties of this present space in a past or a future eternity. He knows, indeed, that the laws assumed by Euclid are true with an accuracy that no direct experiment can approach...; but he knows this as of Here and Now; beyond his range is a There and Then, of which he knows nothing at present but may ultimately come to know more. So, you see, there is a real parallel between the work of Copernicus and his successors on the one hand, and the work of Lobatchewsky and his successors on the other."—*Lectures and Essays,* Vol. I. pp. 298—300.

But in spite of this eulogium, the conception of space which has left the deepest traces in Clifford's writings, is not that of Lobatchewsky, but that founded on the alternative hypothesis (rejected by the Russian geometer) of a constant positive curvature. This conception lies at the bottom of Clifford's theory of biquaternions, to which he devoted much continuous thought, and which was the origin of his researches into the classification of geometric algebras. A space of constant positive curvature is most easily represented to the mathematician (in the absence of any possibility of imaging it to the mind) as the locus of an equation of the form

$$x^2 + y^2 + z^2 + w^2 = \text{constant}$$

in a flat space of four dimensions in which $xyzw$ are rectangular co-ordinates. It is related to the two dimensional surface of a sphere, just as in ordinary geometry space of three dimensions is related to a plane surface. The following description of a space of this kind is taken from the Lecture "On the Postulates of the science of Space." It can hardly be necessary to point out, that in the last sentence Clifford is half laughing at himself.

"I cannot perhaps do better than conclude by describing to you as well as I can what is the nature of things on the supposition that the curvature of all space is uniform and positive.

"In this case the universe, as known, becomes again a valid conception; for the extent of space is a given number of cubic miles. And this comes about in a curious way. If you were to start in any direction whatever and move in that direction in a perfect straight line according to the definition of Leibnitz; after travelling a most prodigious distance, to which the parallactic unit 200,000 times the diameter of the earth's orbit would be only a few steps, you would arrive at—this place.... Upon this supposition of a positive curvature the whole of geometry is far more complete and interesting: the principle of duality, instead of half breaking down over metrical relations, applies to all propositions without exception. In fact, I do not mind confessing that I personally have often found relief from the dreary infinities of homaloidal space in the consoling hope that, after all, this other may be the true state of things."—*Lectures and Essays*, Vol. I. pp. 322—3.

A third line of thought, different from those followed by Lobatchewsky and by Riemann, had no doubt a large share in determining Clifford to regard the hypothesis of constant positive curvature with special favour. One of the earliest geometrical enquiries of wide scope which interested him was the connexion between the descriptive and metrical properties of figures. In two unfinished memoirs "On Analytical Metrics" (XI.), and "On the Theory of Distances" (XVI.), he applied himself to work out the conception, which he justly attributed to Poncelet, that the metrical properties of any figure are in reality descriptive properties of the figure considered in relation to certain fixed geometrical elements, which Professor Cayley has termed the Absolute. In the ordinary geometry of a plane, the Absolute consists of two fixed imaginary points, and of the real straight line containing them, [the imaginary circular points, and the straight line, at an infinite distance].

For these two imaginary points, in the geometry of a spherical surface, we have to substitute the imaginary circle in which the sphere is cut by the plane at an infinite distance. In ordinary space of three dimensions the Absolute is the same imaginary circle and the plane at an infinite distance in which it lies. Professor Cayley in his celebrated "Sixth Memoir on Quantics", generalized this conception by substituting any quadric whatever for the imaginary circle at an infinite distance (which may be regarded as a quadric surface of which one dimension has vanished). The effect of this substitution is to change the metric properties of space, the nature of the change depending on the nature of the quadric chosen as the Absolute. If we wish the space which we thus bring under contemplation to possess one of the most obvious properties which we know by experience to characterize the space in which we live, (viz. that the rotation round a fixed axis, which brings a body from any given position back into the same position again, is a finite and not an infinite operation), our choice of the form of the Absolute is limited to three hypotheses, (1) the Absolute is an imaginary quadric, (2) the Absolute is a real umbilical quadric (*i.e.* a quadric not having real right lines on it) and the space considered is internal to the quadric, (3) the Absolute is an imaginary quadric which has degenerated into a conic section by losing one of its dimensions. Of these three hypotheses the last corresponds to the ordinary conception of space: the spaces characterized by the suppositions (1) and (2) have been termed elliptic and hyperbolic respectively by Professor Klein, who succeeded in showing that in each of them the curvature is constant, being positive in the elliptic, and negative in the hyperbolic space. Thus the geometry of Lobatchewsky is the geometry of hyperbolic space: and Professor Klein's discovery of the identity of the two has thrown a wholly new light upon the researches of the former geometer. Of Clifford's study of the details of the system of Lobatchewsky only one brief note

is preserved (Appendix, p. 531). Indeed he seems to have quickly abandoned hyperbolic for elliptic geometry, influenced no doubt by the reason indicated in the passage which we have quoted—the perfect duality of the properties of elliptic space.

In the geometry of Lobatchewsky every straight line has two real points on it at an infinite distance, viz. the two real points in which it intersects the Absolute. Again, among the planes which pass through a given straight line there are two which belong to the Absolute, and which therefore are to be regarded as planes at an infinite distance. But these two planes are imaginary, being the two planes which can be drawn through the given line to touch the Absolute. Thus in the hyperbolic geometry there is no perfect duality, because when we compare the points which lie along a line, and the planes which pass through it, the absolute elements are real in the one case, and imaginary in the other: in fact, the space which is the dual correlative of an hyperbolic space is not itself a similar space, but is analogous to the space *outside* the Absolute of the hyperbolic space. On the other hand, in elliptic geometry all the elements of the Absolute, whether points or planes, are imaginary, and the duality is as perfect as it is on the surface of a sphere. It follows at the same time that all distances as well as all rotations are finite, and that a point moving on a straight line (or more properly on a shortest line) will come round after a finite journey to the point from which it set out, just as a plane revolving round a straight line returns after describing a finite angle of 360^0 to its original position.

We proceed to give an enumeration of the memoirs contained in this volume, grouped according to their subjects. Perfect accuracy is not important, nor indeed attainable, in such a classification, of which the only object is to convey a general impression of what Clifford has done in each department

of mathematical science. The grouping, however rough, will of itself serve to distinguish the problems with which he occupied himself habitually, and by deliberate preference, from those which had only a temporary interest for him, and were suggested by some accidental circumstance. The enumeration is followed by a few remarks on the methods or results of some of the more important papers.

A. ANALYSIS.

(*a*) *Mathematical Logic.*

(1) On the types of compound statement involving four classes, (I.).

(2) Enumeration of the types of compound statement, (II.).

(*b*) *Theory of Equations and of Elimination.*

(1) Proof that every equation has a root, (IV.).

(2) On a case of Evaporation in the order of a Resultant, (XVII.).

(*c*) *Abelian Integrals and Theta Functions.*

(1) On the Canonical form and Dissection of a Riemann's surface, (XXVII.).

(2) On groups of Periodic Functions, (XXXVIII.).

(3) Theory of Marks of Multiple Theta Functions, (XXXIX.).

(4) Double Theta Functions, (XL.).

(*d*) *Invariants and Covariants.*

(1) Remarks on the Chemico-Algebraical Theory, (XXVIII.).

(2) Notes on Quantics of Alternate Numbers, (XXIX.).

(3) Binary forms of Alternate Variables, (XXXI.).

(*e*) *Miscellaneous.*

(1) Remarks on a Theory of the Exponential Function, (XLV.).

(2) On Bessel's Functions, (XXXVII.).

(3) On a Canonical form of Spherical Harmonics, (XXV.).

(4) A Fragment on Matrices, (XXXV.).

B. GEOMETRY.

(*a*) *Projective and Synthetic Geometry.*

(1) On some Porismatic Problems, (III.).

(2) A Synthetic proof of Miquel's Theorem, (VIII.).

(3) Analogues of Pascal's Theorem, (X.).

(4) Analytical Metrics, (XI.).

(5) On the Theory of Distances, (XVI.).

(6) Geometry on an Ellipsoid, (XIX.).

(7) On the general Theory of Anharmonics, (XII.).

(8) A Geometrical Theorem, (XLVII.). [An early note on the Properties of the Quadrilateral].

(9) Triangular Symmetry, (XLVIII.).

(10) On the Powers of Spheres, (XXXIV.).

(11) Tricircular Sextics, (XXXVI.).

(12) On the Triple Generation of Three Bar Curves, (XLVI. ii.).

(13) On the Mass-centre of an Octahedron, (XLVI. iii.).

(14) On some extensions of the Fundamental Proposition in M. Chasles's Theory of Characteristics, (XLIX.).

(*b*) *Applications of the Higher Algebra to Geometry.*

(1) On Jacobians and Polar Opposites, (VI.).

(2) On a Generalization of the Theory of Polars (XIII.).

(3) On Syzygetic relations among the powers of Linear Quantics, (XIV.).

(4) On Syzygetic relations connecting the powers of Linear Quantics, (XV.).

(5) On a Theorem relating to Polyhedra, (XVIII.).

(6) On Mr Spottiswoode's Contact Problems, (XXXII.)

(*c*) *Geometrical Theory of the Transformation of Elliptic Functions.*

(1) On the Transformation of Elliptic Functions, (XXII.).

(2) Note on the preceding communication, (XXIII.).

(3) On In- and Circum-scribed Polyhedra, (XXIV.).

(*d*) *Kinematics.*

(1) On the principal Axes of a Rigid Body, (VII.).

(2) On a Graphic representation of the Harmonic Components of a Periodic Motion, (XXI.).

(3) On Vortex Motion, (XLVI. i.). [A quaternion solution of a Kinematical Problem].

(4) Instruments used in Measurement.—Instruments illustrating Kinematics, Statics and Dynamics, (L. and LI.). [From the Hand-book to the Special Loan Collection of Scientific Apparatus].

(*e*) *Generalized Conceptions of Space.*

(1) On the Hypotheses which lie at the basis of Geometry, (IX.) [Translation of a discourse of Riemann].

(2) Preliminary Sketch of Biquaternions, (XX.). [Elliptic space].

(3) Further note on Biquaternions, (XLII.). [Elliptic space].

(4) Motion of a Solid in Elliptic space. (XLI.).

(5) On the Theory of Screws in a space of constant positive curvature, (XLIV.). [Elliptic space].

(6) Applications of Grassmann's extensive Algebra, (XXX.).

(7) On the classification of Geometric Algebras, (XLIII.).

(8) On the Free Motion under no forces of a rigid system in an n-fold Homaloid, (XXVI.).

(9) On the Classification of Loci, (XXXIII.).

A (*a*). The paper (I.) on the types of compound statement involving four classes, and the note (II.) upon the general problem of compound statements involving any number of classes, belong to the theory of combinations, which at no time appears to have engaged Clifford's attention continuously. Indeed the only other problems of combinatorial analysis which he has treated are those relating to the marks or systems of indices of the multiple Theta functions.

He appears to have undertaken a discussion of the problem of compound statements rather from an interest casually awakened by its acknowledged difficulty (see the Editor's note, p. 16), than from any predilection for the mathematical theory of Logic, to which (so far as can be gathered from his remaining papers) he never returned again.

A (*b*). The proof given in (IV.) of the fundamental theorem that every equation has a real quadratic factor, is very remarkable because it depends solely on the theory of elimination. A proof substantially the same was afterwards obtained independently by Mr J. C. Malet, of Trinity College, Dublin, and is published in the *Transactions of the Royal Irish Academy* for 1878. The principle of the demonstration is so simple and natural that, when once it has been pointed out, the only wonder is it should not have occurred to any one before. It turns on what Clifford calls an obvious remark that the order of a resultant is reduced when the weight of the coefficients, in the two equations from which the elimination takes place, falls, not by successive units, but by some other constant integer. The development of this remark forms the subject of XVII.

A (*c*). We have already referred to the canonical form of a Riemann's surface imagined by Clifford. His theorem in effect is that a Riemann's surface having n sheets or leaves, and w spiral or branch points, by winding around which the different sheets are connected with and pass into one another, may, if the material of which it is made is sufficiently extensible, be

transformed without tearing into the surface of a body with $\frac{1}{2}w - n + 1$ holes through it. For example, an anchor ring is a body with one hole through it; and into the surface of an anchor ring a Riemann's surface with two sheets and four spiral points can be transformed. Again, in chains of a not uncommon pattern, each link is of the form of a flattened ellipsoid with two circular holes through it; the surface of such a body would represent the equation $y^2 = f(x)$, where $f(x)$ is a rational and integral function of the sixth order, and might therefore be used as the geometrical scaffolding for the theory of hyperelliptic integrals. A closed box with p tubes, open at each end, carried through it from its lid to its bottom, is a simple example of a body with p holes through it. Such a surface would serve to represent (in a certain sense) all the real and imaginary points of a curve of deficiency p; or, which is the same thing, all the pairs of values of two variables which satisfy an algebraical equation of that deficiency. What however is represented in this way is not the quantitative relation between the two variables, but the *connexion* between the different values of one of them corresponding to one and the same given value of the other; viz. if z is the independent and s the dependent variable in an algebraical equation between z and s, and if s_0, s_0' are two different values of s corresponding to the same value z_0 of z, these two values are *connected* in the sense that it is always possible, by making z pass through a continuous cycle of values beginning and ending with z_0, to cause s, though it may have set out with the value s_0, to end with the value s_0'. Each point of any Riemann's surface representing the equation corresponds to a pair of values of z and s; and the different tracks on the surface, irreducible to one another by any continuous change, along which we can pass from the point (z_0, s_0) to the point (z_0, s_0'), answer precisely to the different courses of values by which we can pass with analytical continuity from the pair of values (s_0, z_0) to the pair of values (s_0', z_0). Remodelled as it has been by Clifford, the Riemann's

surface enables us to form a distinct conception of these different tracks, as well as of the systems of curves or "cross-cuts" which have to be drawn in order to reduce the surface to one simply connected; *i.e.* to a surface in which but one set of irreducible tracks can be drawn from any one given point to another. The advantage resulting from Clifford's simplification will be acknowledged by students approaching the theory for the first time, who generally find considerable difficulty in eliciting from Riemann's description a clear image either of the surface itself or of the complicated systems of curves which he directs to be drawn upon it.

The unfinished memoir on the Double Theta Functions (XL.) relates to the problem of the inversion of hyper-elliptic integrals, solved, so far as its main outlines are concerned, by Rosenhain and Goepel. Except in a few matters of detail Clifford does not seem to have added anything to the results of his predecessors, and the papers may be regarded as a continuation of the "Algebraical Introduction to Elliptic Functions" (Appendix, p. 443). The memoir on groups of periodic functions (XXXVIII.) deals with the multiple Theta functions. In it Clifford considers the different Theta functions of the same arguments as multiples by an exponential of any one of them, in which the arguments are increased by *quadrants* (i.e. by multiples of the halves of the periods and of the quasi-periods): and he determines the number of even and uneven Theta functions respectively. Leaving these known results, he proceeds to investigate the differential coefficient of the quotient of two Theta functions (p. 354). The method is the same as that employed in the "Algebraical Introduction;" viz. two Theta series are multiplied together, and the sums of two squares which appear as exponents in the product are transformed by the formula $(m-n)^2+(m+n)^2=2(m^2+n^2)$. But the research is left unfinished, and it does not appear whether Clifford had succeeded in completing it. In the memoir "on the Marks of Multiple Theta Functions" the point of view is somewhat

different, and was evidently suggested by the researches of Riemann and of Dr Weber on the Triple Theta Functions. The expression of a Theta function of multiplicity μ involves μ pairs of indices, each of which is either 0 or 1; thus there are $2^{2\mu}$ different Theta functions of the same arguments, each having its own system of indices, termed by Riemann and Weber the *characteristic*, by Clifford the *mark* of the function. The memoir of Weber[1] contains a complete theory of the marks of the triple Theta functions; and this theory Clifford endeavours to extend to the Theta functions of any multiplicity. What place the fragmentary, but not unimportant, results obtained by him will one day take in a complete theory of the Theta functions must be left to the future to decide. We may observe however that Clifford, in the latest note which we have of his (see p. 329), intimates that he had found the true generalization, for the most general Abelian Integrals, of the beautiful theorem by which Riemann has established a correspondence between the 28 double tangents of a quartic curve and the 28 uneven Theta functions of multiplicity 3. The details of the "Theory of Marks" may perhaps be found dry and repulsive, as indeed is the case with many questions relating to the combinatorial analysis; but the generalization to which we have referred shews what great importance they may hereafter be found to possess for the theory of algebraical integrals.

A (*d*). The study of the geometrical methods of Grassmann, so long neglected even in Germany, made a great and enduring impression on Clifford's mind. Speaking of the *Ausdehnungslehre* in the paper (XXX.) entitled "Applications of Grassmann's Extensive Algebra," he says "I may be permitted to express my profound admiration of that extraordinary work, and my conviction that its principles will exercise a vast influence upon the future of mathematical science."

[1] *Theorie der Abelschen Functionen vom Geschlecht* 3. Berlin, 1876.

A straight line, of which the end-points are A and B, has been, from the days of the ancient mathematicians to our own, universally denoted by the symbol AB. To this notation modern geometry has added the convention $AB = -BA$, which serves to distinguish between the two directions in which the line can be drawn, from A to B, or from B to A. The notations of the Barycentric Calculus of Moebius may perhaps have suggested to Grassmann the idea (which at first sight seems both paradoxical and unpromising) of considering the two points A and B as *factors* of the symbol AB. He was thus led to consider symbolical quantities (which he termed *extensive magnitudes*) $a, b, c, \ldots$, capable of combining with one another by a species of multiplication, subject to the special laws $ab = -ba$, $aa = bb = cc = \ldots = 0$. Such symbolic quantities Clifford, following Dr Sylvester, calls *polar* quantities, and their multiplication *polar* multiplication; the term *alternate* which is certainly less appropriate, had previously been used by him in the same sense. It had been observed (by Grassmann and by Cauchy) that the product of n linear and homogeneous functions of n polar quantities is the determinant of the linear functions multiplied by the product of the polar quantities; *e.g.* $[a\lambda_1 + b\lambda_2] \times [c\lambda_1 + d\lambda_2] = (ad - bc) \times \lambda_1\lambda_2$. And this observation suggested to Clifford the attempt to form invariants and covariants in the same manner. For this purpose he considered in the first place multipartite binary forms in which the indeterminates are polar quantities, and found that by multiplying them together in such a manner that each pair of variables occurs in two factors he obtained an invariant; if, for example, $f = a_{11}\lambda_1\mu_1 + a_{12}\lambda_1\mu_2 + a_{21}\lambda_2\mu_1 + a_{22}\lambda_2\mu_2$, the polar multiplication of f by itself gives

$$f \times f = 2\,(a_{12}\,a_{21} - a_{11}\,a_{22}) \times \lambda_1\lambda_2\,\mu_1\mu_2\,;$$

and this, if we suppose, as we may do, that $\lambda_1\lambda_2 = 1$, $\mu_1\mu_2 = 1$, is the double of the determinant. Invariants obtained in this way have the characteristic property of invariance even when

the different sets of variables are transformed by different linear transformations: but invariants which require that the transformations should be the same may be obtained by a somewhat similar process; *e.g.* if we multiply f by $\lambda_1\mu_2 - \lambda_2\mu_1$ we find the invariant $a_{12} - a_{21}$. In these simple cases the relation of the method to that of "contravariant differentiation" is evident; viz. if in f we write

$$\frac{d}{d\lambda_2}, -\frac{d}{d\lambda_1}; \quad \frac{d}{d\mu_2}, -\frac{d}{d\mu_1}; \text{ for } \lambda_1\lambda_2; \ \mu_1\mu_2;$$

respectively, and apply to f the operator thus obtained, we get the same invariant which arises from the polar multiplication of f by itself. Clifford has not dwelt on the connexion of the two methods; but it must have been present to his mind, since, as we shall presently see, he had already employed the notation of Grassmann to express contravariant differentiation.

Still in the first instance confining himself to multipartite binary forms, Clifford next found ready-made to his hand a geometrical mode of representing the invariants or covariants obtained by polar multiplication. Let a bipartite linear form, such as f, be represented by -o-, a tripartite form by ♢, the number of rays or bonds attached to the small circle denoting the number of pairs of indeterminants, and the direction of the rays being indifferent. Then the symbol o=o will serve to represent $f \times f$; *i.e.* the invariant $a_{12}a_{21} - a_{11}a_{22}$; or, if we suppose that the two small circles represent two different forms, the same symbol will represent their joint invariant. But this symbol is precisely one of the "graphs" which, according to a theory developed by Dr Sylvester and Professor Cayley, may be employed to represent a saturated chemical compound; viz. if the two circles represent two diatomic elements the symbol represents a compound in which the two atomicities are saturated. In this way a parallelism or correspondence is established between the invariants of a system of forms on the one hand and the formulæ of saturated chemical compounds on the other.

In this singularly beautiful series of conceptions it is difficult to say how much belongs to Clifford and how much to Dr Sylvester: the more so because each of them was ready to attribute to the other the larger share. (See the letter to Dr Sylvester entitled "Remarks on the Chemico-algebraical Theory," and the note of Dr Sylvester at p. 257).

A point of considerable interest in itself appears to have had a special attraction for Clifford. "The part," he writes to Dr Sylvester, "of the theory which astonished me most is its application to intergredient variables when the number in a set is greater than three, such as the six coordinates of a line in the case of quaternary forms. When the original variables are regarded as alternate numbers, then intergredients are simply their binary products. Thus by simply multiplying the linear forms representing two planes we get one intergredient form representing their line of intersection, etc."

The theory of the *intervariants* appertaining to intergredient variables has been so little developed in detail that it is a matter of regret that no other record remains of Clifford's meditations on the subject.

It may perhaps be right to point out that simple and beautiful as are the methods of the formation of invariants by polar multiplication, and of their representation by graphs, the application of these methods to individual cases is not exempt from difficulties. One source of these difficulties Clifford indicates when, in the letter just quoted, he says "of course the main thing is to pass from this system of separate variables [in the multipartite binary forms] to that in which the same variables occur to higher orders in the same form, or back again—what you call unravelment."

Thus in the papers XXIX. and XXXI. in which alone we have any detailed applications of the method to the actual determination of invariants and covariants, a large amount of space has to be devoted to the consideration of the distinction between symmetric and unsymmetric forms. The difficulties

of the method are perhaps even more apparent upon an examination of the lithographed fragments to which reference is made in the note on XXXI. The interpretation of each of the graphs recorded in these fragments is not difficult to decipher. But the method does not afford any answer to the question whether the corresponding invariant or covariant is really existent, or is identically evanescent and therefore nugatory. Thus it will be noticed in Dr Sylvester's note on XXVIII., that there is no difficulty in finding the 'algebraical content' of the graph which represents the discriminant of a cubic; but to establish the non-evanescence of this algebraical content, considerations are required which Dr Sylvester supplies from another source.

A (*e*). Of the notes included under this head perhaps one only—that on Bessel's functions—is purely analytical. To these functions Clifford does not appear to have devoted any continuous attention, though he has again considered them from the same point of view in the paper 'on the multiplication of two infinite series.' (Appendix, p. 474). The note on the exponential function is intended to shew that an exponential operator applied to vectors in one line, in a plane, and in space, leads successively to the conceptions of a ratio having a sign, of a complex ratio, and of a quaternion. The note on spherical harmonics points out that Laplace's equation may be interpreted as signifying that the curve represented by any spherical harmonic equated to zero stands in a certain covariant relation to the absolute imaginary circle upon the sphere. Lastly, the fragment on matrices contains (1) a geometrical investigation of the effect of transformation by a matrix of which the determinant vanishes, (2) a geometrical investigation of the condition that two matrices should be commutative in multiplication.

B (*a*). The papers VIII., XLVIII., XI., XVI., XIX. may perhaps be classed together as all turning on the derivation of the metrical from the descriptive properties of figures by the intro-

duction of the imaginary circular points at infinity. Of these XLVIII. (a note from the *Educational Times*), VIII. "The Synthetic Proof of Miquel's Theorem," XIX. "The Geometry on an Ellipsoid," relate to special problems, but are remarkable for their great elegance. The generalization of Miquel's theorem, and the projection of the lines of curvature of an ellipsoid into confocal anallagmatics may be instanced as results which have obtained considerable and well-deserved celebrity. The papers (XI. and XVI.) on "Analytical Metrics," and on "The Theory of Distances," contain systematic attempts to work out the theory of the connexion between metrical and descriptive properties upon the principles of Poncelet and Professor Cayley. Neither of these memoirs is complete; the former, which is of a more elementary character, was probably discontinued by Clifford in favour of the methods adopted in the second, which remained unpublished at the time of his death. This latter paper, in addition to the results which it contains relating to the foci of curves of any order, is remarkable for the method employed in it, which may be described as an extension of the notation of Grassmann so as to include the contravariant differentiation of Dr Sylvester. This paper indeed might without impropriety have been placed in the division B (*b*); and as in one of its sections the absolute is considered as an imaginary conic instead of a pair of imaginary points, it has close relations with the group B (*d*).

Among the other papers under this head which have a character of great generality, we may call attention to the extension of Chasles's theory of characteristics (XLIX.) and to the admirable memoir on the "General Theory of Anharmonics" (XII.), in which the definition of the anharmonic ratio of four points on a line was extended for the first time to systems of points in a plane and in space. Among papers on special problems we may notice one (perhaps more properly belonging to B, *b*) "on the Analogues of Pascal's Theorem," which, dealing chiefly with the case $n = 4$, relates to the figure formed by two

sets of n lines such that $n(n-1)$ of their n^2 intersections lie on a curve of order $n-1$; and that "on some Porismatic Problems," which contains a proof and an extension by the method of correspondences alone, of Poncelet's celebrated porism of the polygon inscribed in one and circumscribed about another conic section. This last paper has an additional interest, because to this porism of Poncelet and to the connexion established by Professor Cayley between it and the theory of Elliptic Functions, we owe Clifford's subsequent researches on the geometrical representation of Elliptic Transformation.

B (*b*). The theory of contravariant differentiation is the leading idea in the paper "on a Generalization of the Theory of Polars," and in the two papers "on the Syzygetic Relations among the Powers of Linear Quantics." The first of these contains the extension (which perhaps was already known) of the theory of the polar curves of a point with regard to a curve of order n: viz. for the point we substitute a curve $f_m(\xi\eta\zeta)$ of the class m $(m<n)$; and operating on $f_n(xyz)$ by

$$f_m\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)$$

we obtain the polar curve of f_m with regard to f_n. For this conception Clifford subsequently provided (as has been already said) a convenient notation by the extension of the meaning of Grassmann's symbols. The note in the Appendix "on the Polar theory of Cubics" may serve as an illustration of the purposes to which he proposed to apply the definitions of this paper in connexion with Grassmann's Geometric Analysis. The geometrical interpretation of the results of the higher algebra was at one time a subject of great interest to him, and I remember his expressing to me a confident expectation that such an interpretation could in all cases be obtained by the combination which he was then employing of the method of contravariant differentiation with that of Grassmann's *Ausdeh-*

nungslehre. How far, at the time when he thus expressed himself, geometers were from having kept pace with the rapid development of algebra, may be inferred from the fact, to which Clebsch often adverted in conversation, that the geometric interpretation of the simplest known linear covariants—those of the binary quintic—has not yet been given. In the two papers "on syzygetic relations" (which do not imply any reference to the methods of Grassmann), Clifford generalizes in various directions the methods employed and the results obtained by M. Paul Serret in his *Géométrie de Direction.*

Among researches of a somewhat more special kind, we must especially mention (1) "The Theorem relating to Polyhedra" (XVIII.) enunciated at p. 169, which is the analogue of an equally beautiful property of polygons discovered by the late T. Cotterill; and (2) the investigations relating to Mr Spottiswoode's problem of "the contact of conics and of quadrics with surfaces."

B (*e*). Passing over the important papers on Elliptic Transformation, the general character of which we have already indicated, and the few notes on kinematical subjects (which are of less importance as Clifford has left a complete work on dynamical science), we come to the class of researches in which he seems to have found the fullest scope for his mathematical imagination.

To the translation of Riemann's discourse we have already referred; and, as Clifford has added no notes of his own to it, we may pass it over here, only observing that the Lecture on the Postulates of the Science of Space is not only a popular account of the theory, but a commentary, which any young mathematician reading Riemann's discourse for the first time will find invaluable. Of the other papers enumerated under this head, all except the last agree in the employment of symbolic methods, founded on those of Grassmann and Sir William Hamilton, but modified and enlarged so as to include within

the domain of symbolic geometry the theory of screws invented by Dr Ball. And it must be allowed that Clifford has succeeded in combining these, in form if not in substance, somewhat heterogeneous elements into a theory of great beauty and wide scope, which is certainly his own. He appears to have shared the conviction of Grassmann* that Sir William Hamilton's quaternions lie entirely within the four corners of the *Ausdehnungslehre*. Some exception may perhaps be taken to this view. In the first place Clifford and Grassmann do not altogether agree as to the way in which the symbols of Grassmann are to be identified with those of Hamilton; and each of them has to assume a law of multiplication for the fundamental units, which is at variance with the law actually adopted in the *Ausdehnungslehre*, though included as a special case under one of the general types of multiplication described by Grassmann in his Memoir "Sur les différents genres de Multiplication" (Crelle's *Journal*, Vol. XLIX. p. 123). In fact, throughout the *Ausdehnungslehre* of 1862 only one species of multiplication (beside that of ordinary algebra) is employed: viz. the polar multiplication (see above, p. lvi), characterized by the equations $[ab] = -[ba]$, $[a^2] = [b^2] = \ldots = 0$, where $a, b, \ldots$ are any extensive magnitudes whatever, and the square brackets are employed to signify that the multiplication is polar and not algebraical. It is true that Grassmann uses a great number of different terms to describe what he considers to be different kinds of multiplication (outer, inner, progressive, regressive, planimetric, stereometric); but these terms really define the interpretation to be given to the symbols in certain combinations, and not the law of the multiplication itself; for example, the *inner* product of an unit a by another unit b is the polar (or, to use Grassmann's expression, the *combinatorial*) product of a, not by b, but by a certain extensive magnitude (viz. the product taken with a definite sign of all the fundamental units

* See his memoir "Der Ort der Hamilton'schen Quaternionen in der Ausdehnungslehre," *Mathematische Annalen*, Vol. XII. p. 375.

which are not factors of b), termed by Grassmann the *complement* of b. To bring quaternions within the methods of the *Ausdehnungslehre*, Grassmann has to suppose a law of multiplication, such that the product of two factors is equal to the complement of their outer or combinatorial product, diminished by their inner product*; and Clifford has to suppose a system of three fundamental units, of which the squares are -1 instead of zero. Each of these suppositions seems to lie wholly outside the particular system of symbolic geometry developed in the *Ausdehnungslehre;* though it is not contended that either of them lies outside the general principles of the memoir "Sur les différents genres de Multiplication." So far as Clifford's own work is concerned, the question of the true relation between the quaternion algebra and that of extensive magnitudes is not perhaps very important: it is certain however that a comparison of the points of difference and of agreement between the two theories exercised a marked influence on the course of his own speculations.

The conception of a vector in space, and of the addition and subtraction of vectors, is common to both systems, and was no doubt familiar to geometers before the invention of either. The idea of a quotient of two vectors is peculiar to the calculus of quaternions; and the expression of such a quotient as the sum of a scalar and a vector is the central point in the whole theory. There is nothing properly corresponding to this in the *Ausdehnungslehre.* We do indeed meet in the work of Grassmann with quotients of vectors†; but these quotients are operators, converting simultaneously three given vectors into three other given vectors; and thus affording a symbolic representation of the homographic transformation of space. Nor again are the quotients of Grassmann themselves extensive magnitudes; thus they do not possess the characteristic property of the vector-quotients

* *Mathematische Annalen*, Vol. XII. p. 378.

† *Ausdehnungslehre*, Arts. 377—390. See also *Math. Ann.* Vol. XII. p. 381—383.

of Sir William Hamilton—that of admitting of expression in terms of the same units as the vectors which are compared. On the other hand, the calculus of quaternions is a calculus of vectors only; *i.e.* it is a geometric calculus in which equal lengths in parallel directions are regarded as equivalent. But the analysis of Grassmann carefully distinguishes between vectors (*Strecken*) and rotors (*Linientheile*), *i.e.* vectors of which the position is limited to some one indefinite straight line; viz. if A and B are two extensive magnitudes denoting simple points, their product $[AB]$ and their difference $B-A$ are respectively the rotor and vector AB. The statical theorems that the sum of any number of rotors can be reduced in an infinite number of ways to the sum of two rotors, and in one way to a rotor and an area of which the axis coincides with that of the rotor, are shewn by Grassmann to follow, upon the principles of his calculus, from the definition of a rotor as the product of two points. Seeing then that rotors as well as vectors could be brought within the range of a geometric calculus, Clifford was naturally led to enquire whether the quotient of two rotors might not be capable of similar expression; and this enquiry led him to the conception of a biquaternion. One step in the way (and an important one) had already been made by Dr Ball's theory of screws. Any composite geometrical quantity consisting of a rotor and of a vector having the same (or any parallel) axis is of the nature of a screw; the pitch of the screw being the quotient of the absolute length of the vector divided by that of the rotor. Such a composite quantity Clifford proposes to call a *motor;* the name being suggested by the statical theorem just referred to, which in fact asserts that any system of forces (*i.e.* of rotors) in space is equivalent to a rotor and a parallel vector (the latter representing an area). It was obvious that the quotient of two rotors is a quantity of the screw or motor type—a tensor-twist: but to complete the symbolic geometry of space an expression for the quotient of two motors had to be obtained; and for this an entirely new conception

had to be invented, that of the Biquaternion*. It is possible that Clifford might not have been guided to this conception if he had not considered the problem with reference to elliptic instead of parabolic space. In an elliptic space a motor can in general be represented in an infinite number of ways as the sum of two rotors, and in one way, and one way only, as the sum of two rotors reciprocal to one another with regard to the absolute. If we represent by ω a rectangular twist of unit pitch, Clifford shews (1) that all such twists are equivalent to one another; (2) that if α be any rotor, $\omega\alpha$ is the rotor reciprocal, and equal to it; (3) that every motor can be expressed in the form $\alpha + \omega\beta$, where α and β are rotors passing through the origin; and (4) that the ratio of any two motors can be exhibited in the form $s + \omega t$ where s and t are quaternions. The symmetry of this system is very striking; in parabolic space the definition of the symbol ω is far less simple and natural (see p. 186)†.

These ideas are developed (though only too briefly) in the "Preliminary Sketch of Biquaternions," the "Further Note on Biquaternions," and the note "On the theory of Screws in a space of Constant Curvature." The paper "On the motion of a body in elliptic space" contains a singularly beautiful application of the whole theory.

* The word Biquaternion had already been employed by Sir W. R. Hamilton. But his biquaternions are entirely different from those of Clifford, and are simply quaternions of which the coefficients are complex quantities containing the imaginary unit of ordinary algebra.

† Dr Ball, who kindly allowed me to submit to him a proof of the above very imperfect account of Clifford's theory of biquaternions, writes to me as follows: "I think it might be well to add a line or two with a view of prominently bringing out Clifford's conception of the 'vector' in elliptic space. A 'right vector' is the result of two equal rotations about a pair of conjugate polars with regard to the absolute; a 'left vector' is the result of two equal *and opposite* rotations about a pair of conjugate polars. Clifford shows beautifully that this is the legitimate generalization of the Hamiltonian vector, and he enunciates the splendid theorem that *any* motor must be *in one way* the sum of a right vector, and a left vector."

Of the papers entitled "Applications of Grassmann's extensive Algebra" and "On the classification of Geometric Algebras," no clearer account can be given than that contained in Clifford's letter to Dr Sylvester: "I had designed for you a series of papers on the application of Grassmann's methods, but there is only one of them fit for printing yet. It is an *explanation* of the laws of quaternions and of my biquaternions by resolving the units into factors having simpler laws of combination; a determination of the compounding systems for space of any number of dimensions; and a proof that the resulting algebra is a compound (in Peirce's sense) of quaternion algebras. It thus appears that quaternions are the last word of geometry in regard to complex algebras."

The fundamental units $i_1, i_2 \ldots i_n$ in these papers, in accordance with the *Ausdehnunglehre,* are points or vectors; vectors being equivalent to points of weight zero at an infinite distance. But these units differ essentially from those of Grassmann, inasmuch as $i_r^2 = \pm 1$, instead of $i_r^2 = 0$, although $i_r i_s = -i_s i_r$. It must be admitted that in the combination of these conditions there is something paradoxical, for the equation $i_r i_s = -i_s i_r$, if we suppose it to hold when $r = s$, gives necessarily $i_r^2 = 0$: and Clifford has nowhere clearly explained why the square of a symbol denoting a point should be a positive or negative unit in the same symbolical system in which the product of two points represents a distance.

The only application of this geometrical algebra which he has left consists in his solution of the problem of the motion of a rigid body round a fixed point in a flat space of n dimensions. The paper (XXVI.) containing this solution is further remarkable as sketching the way in which Theta functions of $n-r$ arguments might be employed to integrate the equations of motion in the case in which no forces are acting; just as Jacobi has expressed as Theta functions of the time the nine direction cosines of a rigid body, rotating under the action of no forces round a fixed point in space of three dimensions.

The last paper under this head, that "on the Classification of Loci," is of a different character from those which have preceded; and is perhaps, even in the unfinished state in which we have it, the most profoundly interesting of all Clifford's mathematical writings. No use is made in it of any special symbolical methods: the only apparatus employed is that of ordinary algebra. And, if the language of the geometry of many dimensions is adopted throughout, yet the results are capable of immediate translation into the language of algebra. Thus the Theorem A, p. 307, "Every proper curve of the nth order is in a flat space of n dimensions or less," is equivalent to the following algebraical proposition:

"If there be a system of $n+m+1$ quantities x connected by $n+m-1$ homogeneous equations; and if this system be such that, upon the addition to it of one equation more, linear and homogeneous in the quantities x and having arbitrary coefficients, it gives n sets of values for the ratios of the quantities x, these $n+m+1$ quantities can always be expressed as linear and homogeneous functions of $n+1$ quantities."

The value of such an algebraical proposition cannot be questioned, because the study of the properties of systems of algebraical equations is of importance for every part of analysis. But the advantage of the geometrical statement in point of clearness and precision is unmistakeable. And our sense of this advantage would be yet further quickened if we were to attempt to render into pure algebra the proof, in three lines only, which Clifford has given of the Theorem. As an example, having no immediate connexion with Clifford's memoir, of the use which has been made by other mathematicians of a similar extension of geometrical language, we may refer to the researches on elimination contained in Lesson XVIII. of the Higher Algebra of Dr Salmon, who cannot be justly accused of having shewn any undue partiality for space of more than three dimensions.

The fundamental principle of Clifford's investigation has been so clearly explained by Professor Henrici in his note on p. 307, that we need not dwell on it here. It may however be worth while to observe that Grassmann (whose footsteps Clifford is certainly not following in this paper) had already proposed to regard the coefficients in the Cartesian equation of a curve as coordinates; and as an example of this method had worked out the geometry of a system of circles in a plane, shewing its analogy to the geometry of the points of space of three dimensions: viz. if $XYZW$ are the equations of four circles, the equation of any given circle may be expressed in the form $lX + mY + nZ + rW$, and l, m, n, r may be regarded as the homogeneous coordinates of this circle. These coordinates of Grassmann's are closely related to the 'power coordinates' of a circle and of a sphere employed in XXXIV. and in the Appendix, p. 546; Clifford's power coordinates of a circle being in fact linear functions of Grassmann's. It is remarkable, however, that Grassmann, at least in his general definition, contemplates only a single equation between his generalized coordinates. Thus the circles of which the coordinates satisfy the homogeneous relation $f(l, m, n, r) = 0$, form the only "Kurvengebilde" of circles which he defines. This according to Clifford would be a "two spread" of circles, and a "one spread" or "one way locus" would be formed by the circles of which the coordinates satisfy two equations. All that remains to us of Clifford's work on the subject relates to "one-way loci."

The application of Abelian functions to this new aspect of geometry awakened all Clifford's enthusiasm. He spoke to me of this part of his theory as opening a boundless field for new researches—as "altogether too big a thing" for one man to manage; and, with the instinct for companionship so characteristic of his nature, he expressed an earnest desire to get others to join him in the work. All that remains to indicate the nature of the discoveries which he had made in this direction is (1) a brief but masterly summary of the known results of the

theory of Abelian functions, stated in a form suitable to the applications which he contemplated; (2) a generalization, to which we have already referred, of an important result obtained by Riemann; and (3) a remarkable theorem limiting, in certain cases, the number of dimensions requisite for the existence of a curve of given order and deficiency. How much may have perished unrecorded we cannot tell. But, however this may be, no geometer will look for a more splendid monument of Clifford's genius, or for a more touching memorial of his early death, than is to be found in the unfinished pages 'On the Classification of Loci' which embody the last and perhaps the greatest effort of his inventive powers.

MATHEMATICAL PAPERS.

I.

ON THE TYPES OF COMPOUND STATEMENT INVOLVING FOUR CLASSES*.

PROFESSOR STANLEY JEVONS has enumerated† the types of compound statement involving three classes, among which the premises of a syllogism appear as a type of four-fold statement. He propounded at the same time the corresponding problem of enumeration for four classes, which is solved in the present communication. The reader is referred to the paper or the book just mentioned for further explanation of the nature and purpose of the problem than is to be found in Art. 1. It may, however, be premised that the letters A, B, C, D, denote four *classes* or *terms* (for example, hard, wet, black, nice), and that, according to a convenient notation of De Morgan's, the small letters, a, b, c, d denote the complementary classes or contrary terms (not hard, not wet, not black, not nice). A *simple* statement is of the form $ABCD = 0$ (no hard, wet, black, nice things exist, or, which is the same thing, all hard, wet, black things are nasty). The statement $ABC = 0$ (no hard, wet, black things exist, or all hard, black things are dry) is to be regarded as made of these two, $ABCD = 0$, $ABCd = 0$ (no hard, wet, black, nice things exist, and no hard, wet, black, nasty things exist) and so is called a *compound* (in this case a *two-fold*) statement. The notion of *types* is defined in Art. 1.

1. Four classes, or terms, A, B, C, D, give rise to sixteen cross-divisions or *marks*, such as $AbCd$. A denial of the existence of one of these cross-divisions, or of anything having its mark (such as $AbCd = 0$), is called a simple statement. A

* [From the *Memoirs of the Literary and Philosophical Society of Manchester*, Session 1876—77. Vol. xvi. No. 7, pp. 88—101. Communicated January 9th, 1877.]

† *Proceedings of the Manchester Philosophical Society*, vol. vi. pp. 65—68, and *Memoirs*, Third Series, vol. v. pp. 119—130. *The Principles of Science*, vol. i. pp. 154—164. [New Edition, pp. 134—143.]

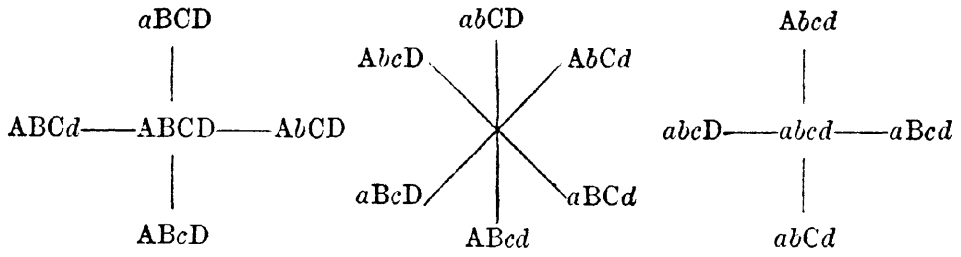
denial of two or more cross-divisions is called a compound statement, and, moreover, two-fold, three-fold, etc., according to the number denied.

When two compound statements can be converted into one another by interchange of the classes A, B, C, D with each other or with their complementary classes *a*, *b*, *c*, *d*, they are called *similar;* and all similar statements are said to belong to the same *type*. The problem before us is to enumerate all the types of compound statement that can be made with four terms.

2. Two statements are called *complementary* when they deny between them all the sixteen marks without both denying any mark; or, which is the same thing, when each denies just those marks which the other permits to exist. It is obvious that when two statements are similar, the complementary statements will also be similar; and, consequently, for every type of n-fold statement there is a *complementary type* of $(16-n)$ fold statement. It follows that we need only enumerate the types as far as the eighth order; for the types of more-than-eight-fold statement will already have been given as complementary to types of lower orders. Every eight-fold statement is complementary to an eight-fold statement; but these are not necessarily of the same type.

3. One mark ABCD may be converted into another A*b*C*d* by interchanging one or more of the classes A, B, C, D with its complementary class. The number of such changes is called the *distance* of the two marks. Thus in the example given the distance is 2. In two similar compound statements the distances of the marks denied must be the same; but it does not follow that when all the distances are the same, the two statements are similar. There is, however, as we shall see, only one example of two dissimilar statements having the same distances. When the distance is 4, the two marks are said to be *obverse* to one another, and the statements denying them are called *obverse* statements—as ABCD, *abcd*; or, again, A*b*C*d*, *a*B*c*D. When any one mark is given (called the *origin*), all the others may be grouped in respect of their relations to it as follows:—Four are at distance *one* from it, and may be called *proximates;* six at distance *two*, and may be called *mediates;*

four at distance *three*, and may be called *ultimates*. Finally, the obverse is at distance *four*.



Origin and 4 proximates. 6 mediates. Obverse and 4 ultimates.

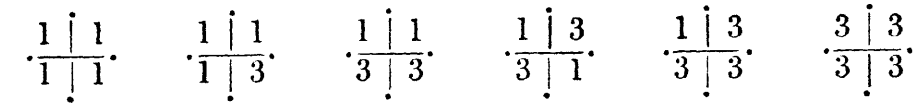
It will be seen from the above table that the four proximates are respectively obverse to the four ultimates, and that the mediates form three pairs of obverses. Every proximate or ultimate is distant 1 and 3 respectively from such a pair of mediates. Thus each proximate or ultimate divides the mediates into two classes; three of them are at distance 1 from it, and three at distance 3. Two mediates which are not obverse are at distance 2. Two proximates, or two ultimates, or an ultimate and a proximate which are not obverse, are also at distance 2.

This view of the mutual relations of the marks is the basis of the following enumeration of types.

4. There is clearly only one type of simple statement. But of two-fold statements there are four types; viz. the distance may be 1, 2, 3, or 4; and so, in general, with n classes there are n types of two-fold statement.

5. A compound statement containing no pair of obverses is called *pure*. In a three-fold statement there are three distances; one of these must be not less than either of the others. If this be 2, the remaining mark must be at odd distance from both of these or at even distance from both; thus we get the types 1, 1, 2, and 2, 2, 2. If the not-less distance be 3, the remaining distances must be one even and the other odd; the even distance must be 2, the odd one either 1 or 3; and the types are 1, 2, 3; 2, 3, 3. Thus there are 4 pure three-fold types. With a pair of obverses, the remaining mark must be at odd or even distance from them; 1, 3, 4; 2, 2, 4. In all six three-fold types observe that there is necessarily one even distance.

6. *A fortiori*, in a four-fold statement there must be one even distance. In a pure four-fold statement this distance is 2. From this pair of marks let both the others be oddly distant; then they must be evenly distant from one another, *i.e.* at distance 2, obverses being excluded. The odd distances are 1 or 3; and it will be easily seen that the following are all the possible cases:



In these figures the dots indicate the four marks, the cross lines indicate distance 2, and the other figures the distances between the marks on either side of them. Next, from the pairs of marks at distance 2 let one of the others at least be evenly distant, *i.e.* at distance 2. Then we have three marks which are all at distance 2 from one another; and it is easy to show that they are all proximates of a certain other mark. For, select one of them as origin; then the other two are mediates which are not obverse, and which consequently are at distance 1 from some one proximate. With this proximate as origin, therefore, all three are proximates. We have therefore only to enquire what different relations the fourth mark can bear to these three. It may be the origin, its obverse, the remaining proximate, its obverse, or one of two kinds of mediates, viz. at distance 1 or 3 from the remaining proximate. Thus we have 6 types, in which the distances of the fourth mark from the triad are respectively 111, 333, 222, 222, 133, 113. The third and fourth of these are especially interesting, as being distinct types with the same set of distances; I call them *proper* and *improper groups* respectively: viz., a proper group is the four proximates of any origin; an improper group is three proximates with the obverse of the fourth. On the whole we get 12 types of pure four-fold statement.

7. In a four-fold statement with *one* pair of obverses, take one of them for origin; the remaining two marks must then be either a pair of proximates or ultimates, a proximate and an ultimate, a pair of mediates, or a proximate or ultimate with one of two kinds of mediate—in all, 5 types, with the distances

13^2,13; 13^2,31; 22^2,22; 13^1,22; 13^3,22. With *two* pairs of obverses they must be either at odd or even distances from one another; two types. Altogether $12 + 5 + 2 = 19$ four-fold types.

8. In a *pure* five-fold statement there is always a triad of marks at distance 2 from one another. For there is a pair evenly distant; if there is not another mark evenly distant from these, the remaining three are all oddly distant, and therefore evenly distant from one another. First, then, let the remaining two marks be both oddly distant from the triad. In regard to the origin of which these are proximates, the two to be added must be either two mediates, like (of two kinds) or unlike, or a mediate of either kind with the origin or the obverse; 7 types. Next, if one of the two marks be evenly distant from the triad, it must form with the triad either a proper or an improper group of four. To a proper group we may add the origin, the obverse, or a mediate; to an improper group, the origin or the obverse (the mediates give no new type), 5 types; or in all 12 pure five-fold types.

9. In a five-fold statement with one pair of obverses there must be another pair of marks at distance 2. We have therefore to add one mark to each of the following three types of fourfold statement,—a pair of obverses together with (1) two proximates, (2) a proximate and an ultimate, (3) two mediates. To the first we may add another proximate, an ultimate, or a mediate of three kinds, viz. at distances 11, 13, 33 from the two proximates; 5 types. To the second if we add a proximate or an ultimate, we fall back on one of the previous cases; but there are again three kinds of mediates, at distances 11, 33, 13 from the proximate and ultimate; 3 types. To the third we may add another mediate, whereby the type becomes a proper group together with the obverse of one of its marks, which is the same thing as an improper group together with the obverse of one of its marks—or a proximate or ultimate which are of three kinds, at distances 11, 13, 33 from the two mediates; 4 types. Thus there are 12 five-fold types with one pair of obverses. With two pairs of obverses at odd distances, there is only one type, all the remaining marks

being similarly related to them; at even distance the remaining mark may be evenly or oddly distant from them; 2 types. On the whole we have $12 + 12 + 3 = 27$ types of five-fold statement.

It is to be remarked that there is no pure five-fold statement in which all the distances are even, and that, if there is only one pair of obverses with all the distances even, the type is a proper group together with the obverse of one of its marks.

10. We may now prove, as a consequence of the last remark, that a pure six-fold statement either contains a group of four with a pair oddly distant from it or consists of two triads oddly distant from one another. For there must be a pair at distance 2; if the other four are all oddly distant from these, they form a group; if one is evenly distant and three oddly distant, we have the case of the two triads; if two are evenly distant, we again have a group. We must add, then, first to a proper group, and then to an improper group, a pair oddly distant from it. To a proper group consisting of the proximates to a certain origin we may add the origin or its obverse with a mediate, or two mediates; 3 types. An improper group is symmetrical; that is to say, if we substitute for any one of its marks the obverse of that mark, we shall obtain a proper group. In this way we shall get four origins distant 1113 from the group, and four obverses distant 1333; if we add to these the obverses of the marks in the group itself, we have described the relation of the twelve remaining marks to the group. To form, therefore, a pure six-fold statement we may add either two origins or two obverses or an origin and an obverse; 3 types.

In the case of the two triads, since they are oddly distant from one another their origins must be oddly distant; that is, they must be distant either 1 or 3. If they are distant 1, neither, both, or one of the origins may appear in the statement; if they are distant 3, neither, both, or one of the obverses; 6 types. Thus we obtain 12 types of purely six-fold statement.

11. If a six-fold statement contains one pair of obverses, the remaining four marks cannot all be evenly distant from this pair. For in that case they would constitute a group; and it is easy to see that the marks evenly distant from a group, whether proper or improper, do not contain a pair of obverses. We have therefore only these four cases to consider:—

(1) The four marks are all oddly distant from the obverses.

(2) One is evenly distant and three oddly distant.

(3) Two are evenly distant and two oddly.

(4) Three are evenly distant and one oddly.

In the first case the four marks form a group. If this is a proper group, the pair of obverses must be either the origin and obverse of the group, or a pair of mediates; 2 types. If the group is improper, the pair must be an origin and an obverse; 1 type. In the second case, we have an origin, an obverse, and a mediate, to which we must add three marks taken out of the proximates and ultimates. We may add 3 proximates distant respectively 113 or 133 from the mediates (2 types),—or 2 proximates distant respectively 11, 13, 33 from the mediate, and with each of these combinations an ultimate distant either 1 or 3 (6 types). To interchange proximates with ultimates clearly makes no difference; so that in reckoning the cases of 1 proximate and 2 ultimates or 3 ultimates, we should find no new types. In the third case we have an origin, an obverse, and two mediates distant 2 from each other; and to these we have to add either two proximates or a proximate and an ultimate. The two proximates may be distant from the two mediates 11, 13, or 11, 33, or 13, 13, or 13, 33; 4 types. The proximate or ultimate must not be respectively distant 11, 33, or 33, 11; for then they would form a pair of obverses; there remain the cases 11 with 11 or 13, 13 with 13, and 33 with 13 or 33; 5 types. In the fourth case we have an origin, obverse, and three mediates distant 2 from one another; the remaining mark must be distant either 1 or 3 from these mediates; 2 types. This makes twenty-two types of six-fold statement with one pair of obverses.

12. If a six-fold statement contains two pairs of obverses, these must be either evenly or oddly distant. If they are evenly distant we have an origin, obverse and two obverse mediates, to which two other marks are to be added. These may be both evenly distant; taking one of them as origin, it is associated with 5 mediates, so that there is 1 type only. Or both oddly distant; here there are two cases, according as the distances are 11, 33 or 13, 13. Or one oddly and one evenly distant; the latter is any one of the four remaining mediates, and then the former is distant 1 or 3 from it; 2 types. If the two pairs of obverses be oddly distant they form an aggregate which is related in the same way to all the remaining twelve marks; viz. any one of these being taken as origin, we have a pair of mediates and a proximate with its obverse ultimate. The thing to be considered, therefore, is the distance between the two marks to be added, which may be 1, 2 or 3, and each in two ways; 6 types.

A six-fold statement with three pairs of obverses is one of two types only; viz. these are all evenly distant when they are the mediates to one origin, or two evenly distant and one oddly distant from both of them.

13. A pure seven-fold statement must consist of a group and a triad; for it must contain a triad, by the same reasoning by which this was proved for a five-fold statement; and then either all the other four marks are oddly distant from this, and so form a group by themselves, or else one of them is evenly distant from the triad and so forms a group with it. If the group is proper, being the proximates to a certain origin, the triad must consist of two mediates and either the origin, the obverse or another mediate; and in the latter case the three mediates are distant 111 or 333 from some proximate; 4 types. If the group is improper, the triad is either all origins or all obverses, or two origins and an obverse, or an origin and two obverses; 4 types. In all, 8 types of pure seven-fold statement.

14. A seven-fold statement with one pair of obverses must consist either of four marks evenly distant from one another

and three oddly distant from them; or of five marks evenly distant from one another and two oddly distant from them. In the former case the pair of obverses may be in the four or in the three. If they are in the four, the three form a triad which are proximates to one origin; and then the pair may be the origin and obverse or a pair of mediates. If the pair are origin and obverse, the other two (at distance 2) are mediates, distant 11, 13 or 33 from the proximate which is not in the triad; if the pair are mediates, the two may be the origin or obverse with a mediate distant 1 or 3 from that proximate (4 types) or two mediates distant 11, 13, 33 from it (3 types). If the pair of obverses are in the set of three marks, the four form a group, which may be proper or improper. If proper, the three may be origin and obverse with a mediate, or a pair of mediates with origin, obverse, or another mediate; 4 types. If improper, the three must be two origins and an obverse, or an origin and two obverses; 3 types.

Five marks evenly distant containing only one pair of obverses, must be a proper group with the obverse of one of its marks; see end of Art. 9. To these we may add the origin or obverse of the proper group with a mediate distant 1 or 3 from the extra mark, or else two mediates distant 11, 13 or 33 from that mark; 7 types.

15. A seven-fold statement with two pairs of obverses may have six marks evenly distant from one another and one oddly distant from them; in this case the six are an origin and five mediates in two different ways, or say two pairs and a two; the remaining mark may be distant 11, 13 or 33 from the two, which gives 3 types.

Otherwise the seven-fold statement must subdivide (as in the last case) into five and two or into four and three. If it subdivide into five and two, the two may be a pair or not. In the first case we have a proper group and the obverse of one of its marks, together with the origin and obverse of the group or a pair of mediates; 2 types. In the second case we have five mediates of an origin or its obverse, to which we may add two proximates distant 11, 13 or 33 from the odd mediate,

or a proximate and an ultimate distant 11, 13 or 33 respectively from the odd mediate; 6 types.

If the seven-fold statement subdivide into four and three, the two pairs may be both in the four, or one in the four and one in the three. In the former case we have a triad, to which may be added the origin and obverse and a pair of mediates, or two pairs of mediates; 2 types. In the latter case the four consist of an origin and obverse and two mediates; we must add a pair consisting of a proximate and an ultimate, which may be distant 11, 33 or 13, 13 from the two mediates, and then another proximate or ultimate which may be distant 11, 13, or 33 from the two mediates; 6 types.

16. Three pairs of obverses in a seven-fold statement may be all evenly distant, or two evenly and the other pair oddly distant from each. If they are all evenly distant they are the mediates to a certain origin or its obverse, and the seventh mark may be the origin or a proximate; 2 types. In the other case we have an origin, obverse, and pair of mediates, together with a proximate and its obverse ultimate; we may add a proximate or a mediate; 2 types.

17. A pure eight-fold statement must consist of two groups, either both proper or both improper, or one of each. Two proper groups may have their origins distant 1 or 3; 2 types. To an improper group we may add a proper group made of one origin and three obverses, or of three origins and one obverse, or an improper group made of four origins or four obverses, or two origins and two obverses; 5 types. Altogether there are 7 types of pure eight-fold statement.

18. An eight-fold statement with one pair of obverses must subdivide into four and four, or into five and three. In the former case we have a pair of obverses, viz. an origin and its obverse, and two mediates; to which we must add a group formed out of the proximates and ultimates. This group may be proper, (1 type,) or improper, the mediates being in regard to it two origins, two obverses, or an origin and an obverse; 3 types. In the latter case the five marks must be a proper group with the obverse of one mark, to which we must add a triad made out of the origin, obverse, and mediates of the

group. This triad may be the origin or obverse together with two mediates distant 11, 13, 33 from the ultimate; 6 types; or else it may be three mediates distant 111, 113, 133, 333 from the ultimate; 4 types.

19. An eight-fold statement with two pairs of obverses must subdivide into four and four, or into five and three, or into six and two. In the first case the two pairs of obverses may be evenly distant, when the remaining marks form a group either proper, with its origin, obverse, and pair of mediates, or two pairs of mediates, or else improper; 3 types; or oddly distant, when the remainder form one of the six pure four-fold statements enumerated Art. 6. Two marks distant 2 from each other may be distant 11, 33 or 13, 13 from the pair of obverses which are oddly distant from them; thus each of the six four-fold statements gives 3 types of eight-fold statement, except the third, which gives 4; in all, 19. In the second case the three may be a triad, or may contain a pair of obverses. If it is a triad, the five are mediates to one origin and its obverse, and we may add three proximates distant 113 or 133 or two proximates distant 11, 13 or 33, with an ultimate distant respectively 11 or 33 from the odd mediate; 6 types. If the three contain a pair of obverses, the five make a proper group with obverse of one mark; to this we may add the origin and obverse of the group with mediate distant 1 or 3 from the ultimate, or a pair of obverse mediates with a mediate distant 1 or 3 as before; 4 types. In the third case the six must be an origin and five mediates, and we may add two proximates distant 11, 13, 33 from the odd mediate, or a proximate and an ultimate, or two ultimates, distant as before; 9 types.

20. In an eight-fold statement with three pairs of obverses these may be either all evenly distant, or two of them evenly distant and the other oddly distant from both. In the first case they are mediates to a certain origin and its obverse, and we may add the origin with a proximate or ultimate, two proximates, or a proximate and ultimate; 4 types. In the second case take the oddly distant pair for origin and obverse; then these are associated with two proximates and their ob-

verse ultimates, and we may add the two other proximates, a proximate and an ultimate, a proximate and a mediate (distant 11, 13, 31, 33 from this proximate and the remaining one), or two mediates distant 11, 33 or 13, 13 from the two proximates; 8 types.

Lastly, in an eight-fold statement with four pairs of obverses they may be all evenly distant, or the statement may subdivide into six and two, or into four and four; in the latter case there are 2 types.

21. To obtain the whole number of types, we observe that for every less-than-eight-fold type there is a complementary more-than-eight-fold type (Art. 2); so that we must add the number of eight-fold types (78), to twice the number of less-than-eight-fold types (159); the result is 396.

TABLE.

Art.				
4	1-fold			1
	2-fold, distance 1, 2, 3, 4			4
5	3-fold, pure, distance 112, 222, 123, 233		4	
	1 pair obv., dist., 134, 224		2	
			6	6
6	4-fold, pure, two and two—			
	$\begin{smallmatrix}1&1\\1&1\end{smallmatrix}$ $\begin{smallmatrix}1&1\\1&3\end{smallmatrix}$ $\begin{smallmatrix}1&1\\3&3\end{smallmatrix}$ $\begin{smallmatrix}1&3\\3&1\end{smallmatrix}$ $\begin{smallmatrix}1&3\\3&3\end{smallmatrix}$ $\begin{smallmatrix}3&3\\3&3\end{smallmatrix}$	6		
	three and one	4		
	group, proper or improper	2		
		12	12	
7	,, 1 pair obv.		5	
	,, 2 pair obv., dist. odd or even		2	
			19	19
8	5-fold, pure, three and two	7		
	four and one	5		
			12	
9	,, 1 pair obv. + two prox.	5		
	+ prox. + ult.	3		
	+ two med.	4		
		12		
			12	
	2 pair obv., odd dist., 1; even, 2		3	
			27	27

TABLE (*continued*).

Art.				
10	6-fold, pure, three and three	6		
	four and two	6		
		12		
		—	12	
11	,, 1 pair obv., two and four	3		
	three and three	8		
	four and two	9		
	five and one	2		
		22		
		—	22	
12	,, 2 pair obv., odd dist., 6; even, 5		11	
	3 pair obv.		2	
			47	47
13	7-fold, pure; proper group, 4; improper, 4		8	
14	,, 1 pair obv., four and three	10		
	three and four	7		
	five and two	7		
		24		
		—	24	
15	,, 2 pair obv., six and one	3		
	five and two	8		
	four and three	8		
		19		
		—	19	
16	,, 3 pair obv.		4	
			55	55
	Total of less-than-eight-fold statements ...		...	159
	Complementary more-than-eight-fold statements		...	159
17	8-fold, pure		7	
18	,, 1 pair obv., four and four	4		
	five and three	10		
		14		
		—	14	
19	,, 2 pair obv., four and four	22		
	five and three	10		
	six and two	9		
		41		
		—	41	
20	,, 3 pair obv., all evenly dist.	4		
	two evenly dist.	8		
		12		
		—	12	
	4 pair obv.		4	
			78	78
	Grand Total		...	396

* II.

ENUMERATION OF THE TYPES OF COMPOUND STATEMENTS.

A SET of n classes A, B, C, ... involve a number 2^n of cross-divisions A, b, C, d, ... etc.; these shall be called simply *divisions*. A *statement* about these classes affirms the non-existence of one or more divisions; and it is called a one-fold, two-fold, etc., statement, according to the number of divisions whose existence it denies. A one-fold statement about a set of n classes is a 2^m-fold statement about a set of these and m other classes. Two statements are called *inverse* when each denies the divisions whose existence is allowed by the other; thus the inverse of a p-fold statement about n classes is a $(2^n - p)$ fold statement.

One division may be converted into another by the operation of changing certain classes into their complementary classes (A into a which is not-A). The number of these changes required is called the *distance* of the two divisions. Thus in four classes, the division $AbCD$ is distant two changes from the division $abcD$.

Two statements are said to be of the same type when one may be obtained from the other by a repetition of two processes: (1) the simultaneous substitution of a class and its complementary for another class and its complementary (A for B and a for b), (2) the interchange of a class and its complementary (A for a). It is clear that this amounts only to a use of new symbols for the classes involved.

If the same substitutions are performed upon any two divisions their distance will remain unaltered. Hence *if two statements are of the same type the relative distances of the divisions denied by them will be the same;* but it does not follow that whenever these distances are the same the statements are of the same type.

The whole number of statements that can be made about n classes is $2^{2^n} - 2$, when we exclude the absence of any statement and its inverse the denial of the universe. The problem of enumeration consists in the distribution of these under a finite number of types; it requires an exhaustive method of describing the types, and may be checked by counting the number of statements belonging to each.

There is only one type of one-fold statement about n classes, namely, $ABC...N = 0$; and as the equation of any division to zero gives a statement of this type it follows that there are 2^n such statements.

There are n types of two-fold statement; for the two divisions equated to zero may have any distance from 1 to n. If this distance is r, the number of statements included in the type is $2^{n-1} \frac{\lfloor n}{\underline{r} \mid \underline{n-r}}$. The whole number of two-fold statements is of course $2^{n-1}(2^n - 1)$.

To determine the number of types of three-fold statements, let α, β, γ be the three divisions denied by such a statement, and suppose the distance $\beta\gamma$ to be not less than either of the distances $\alpha\beta$, $\alpha\gamma$. Let $\alpha\beta = r$, $\alpha\gamma = s$, and let t changes be common to the r and the s; then $\beta\gamma = r + s - 2t$. Let r not be less than s; we have $r + s - 2t \not> r$, therefore $s \not> 2t$, or t must not be greater than half the least of the numbers r, s. With the distances r, s then we shall have distinct types for the values of $t = 0, 1, 2, \ldots I\left(\frac{s}{2}\right)$ (I meaning *integral part of*); that is $I + I\left(\frac{s}{2}\right)$ types, provided that $r + s$ is not greater than

n; in this case t cannot be less than $r+s-n$. First let r be even and equal to $2p$, where $4p$ is not greater than n. Then the number of types is

$$1+2+2+3+3+\ldots+p+p+\overline{p+1}$$
$$=p+p\,(p+1)=p\,(p+2)$$
$$=\tfrac{1}{2}\,r\,(r+1),\quad 2r\not>n.$$

Next let $r=2p+1$, $2r\not>n$; then the

* * * * *

[Prof. Cayley remarks that Prof. Clifford considers, in this fragment, the further question, how many are the statements of each type. In the case of 4 classes, the results up to the 3-fold statement, see Table, p. 12, are as follows:—

	No. of Statements.	
1-fold	16	
		16
2-fold, dist. 1	32	
,, 2	48	
,, 3	32	
,, 4	8	
		120
3-fold, dist. 112	96	
,, 222	64	
,, 123	192	
,, 233	96	
,, 134	64	
,, 224	48	
		560

where the totals 16, 120, 560, &c. are, of course, the numbers for the combinations of 16 things 1, 2, 3, &c. together.

Prof. Jevons in *The Principles of Science* (Third Edition, p. 143) writes: "In the first edition (vol. I. p. 163), I asserted that some years of labour would be required to ascertain even the precise number of types of law governing the combination of four classes of things. Though I still believe that some years' labour would be required to work out the types themselves, it is clearly a mistake to suppose that the *numbers* of such types cannot be calculated with a reasonable amount of labour, Professor W. K. Clifford having actually accomplished the task." It was on reading Prof. Jevons's original statements that Prof. Clifford said laughingly, he couldn't let anyone say that and not do it straight off...it was 'his luck' to do quickly difficult things which were usually got at by long processes only.

Dr Hopkinson has drawn my attention to the fact that the problems treated of by Clifford and Jevons are different in this respect, that Clifford's result includes what Jevons calls 'inconsistent statements'.]

III.

ON SOME PORISMATIC PROBLEMS *.

THE PROBLEM :—To draw a polygon of a given number of sides, all whose vertices shall lie on one given conic, and all whose sides shall touch another given conic : is either not possible at all, or possible in an infinity of ways. This remark, originally made by Poncelet, has been shewn by Professor Cayley to depend in a very beautiful manner upon the theory of elliptic functions; and in this way he has proved that an analogous theorem holds good wherever a (2, 2) correspondence exists: that is to say, whenever two things are so related that to every position of either there correspond two positions of the other. Two points, x, y for instance, in a conic U, which are connected by the relation that the line xy touches a second conic V, have a correspondence of this kind : for if the point x be taken arbitrarily, two tangents can be drawn from it to V, determining two positions of y: and conversely, the point y being fixed determines two positions of x. The theorem is then that in a (2, 2) correspondence there is either no closed cycle of a given order, or an infinite number. In the present communication I propose first to prove this result by the method of correspondence alone, and then to extend the proof to higher orders of correspondence.

In a (2, 2) correspondence there are $4 (= 2 + 2)$ united points, that is to say, four points, each of which coincides with one of its correspondents. In fact, if two numbers x and y are

* [From *Cambridge Philosophical Society's Proceedings*, II. 1876. Read Nov. 9, 1868, pp. 120—123.]

connected by an equation of the second degree in each of them, then when we make x and y coincide, there results an equation of the fourth degree (Chasles, *Comptes Rendus*, 1864). I call these united points the points a. Each point a has one of its correspondents coinciding with it; it has also another correspondent b. Each point b again has another correspondent c, and so on. There are also four points α, each of which is such that its two correspondents coincide in a point β. For let q be a correspondent of p, and r a correspondent of q; then the relation between p and r is a (2, 2) correspondence (since to each position of p there are two positions of r and *vice versa*), and therefore has four united points, viz. the points β. Each of these points β has another correspondent γ, and so on. We have thus two series of points, $abcd\ldots$ $\alpha\beta\gamma\delta\ldots$ each letter indicating a set of four generally distinct points.

Let us now endeavour to obtain a closed cycle of an odd order: for distinctness' sake we will try to draw a pentagon inscribed in one conic, U, and circumscribed to another, V*. Start with a point x on the outer; pass to one of its correspondents, y; y has another correspondent, z; from z we go to u, from u to v, from v to w. If now w were the same point as x, we should have succeeded in our object. But the relation between w and x is a (2, 2) correspondence, for we might have started from x in either of two directions. The united points of this correspondence should therefore apparently give solutions of our problem.

But these united points are no other than the four points c. For starting with one of these, we get the cycle *cbaabc*, which is a sufficient solution of the correspondence problem last enunciated. But it is *not* a solution of the original problem: for the series will go on *cbaabcde*... and not repeat itself, so that the points *cbaab* do not form a *proper* in-and-circumscribed pentagon. Thus the problem is in general impossible. If however there is any proper solution, the equation of the fourth degree

* [Supposing as before that the corresponding points x, y in the conic U are such that the line xy touches the conic V, then, as is easily seen, the points a are in fact the points of contact with U of the common tangents of U and V; and the points α are the points of intersection of U and V.—C.]

(which determines the improper solutions) will have more than four roots, and will therefore be identically satisfied by any number whatever; so that whatever point x we start with, the point w will come to coincide with it.

Precisely similar reasoning is applicable to the cycles of an even order. Thus, *e.g.* for a quadrilateral we get the four improper solutions $\gamma\beta\alpha\beta$, got by starting from the points γ. I pass to the consideration of correspondences of higher orders.

In an (r, r) correspondence there are

$$2r \text{ united points } a;$$

their remaining correspondents form

$$2r(r-1) \text{ points } b;$$

to these again correspond

$$2r(r-1)^2 \text{ points } c, \text{ and so on.}$$

Similarly, there are

$$2r(r-1) \text{ points } \alpha,$$

each of which is such that two of its correspondents coincide; viz. these are

$$2r(r-1) \text{ points } \beta,$$

to which also correspond

$$2r(r-1)^2 \text{ points } \gamma, \text{ and so on.}$$

Now if we attempt to form a closed cycle of the nth order, we shall be led to a correspondence

$$\{r(r-1)^{n-1},\quad r(r-1)^{n-1}\},$$

which has $2r(r-1)^{n-1}$ united points. From this number we shall have to subtract the number of improper solutions as given by our previous reasoning; thus we shall find

$(n = 2m+1)$, $\{2r(r-1)^{2m} - 2r(r-1)^m\}$ proper solutions,

$(n = 2m)$, $\{2r(r-1)^{2m-1} - 2r(r-1)^m\}$ proper solutions.

For example, the problem to inscribe in a conic a triangle whose sides shall touch a given curve of the third class admits of twelve proper and twelve improper solutions. If the number of proper solutions exceeds this number, the problem becomes porismatic: that is to say, there is an infinite number of solutions.

IV.

PROOF THAT EVERY RATIONAL EQUATION HAS A ROOT*.

(*Abstract.*)

THE proof contained in the present communication depends on the determination of a quadratic factor of the rational integral expression

$$x^{2s} + a_1 x^{2s-1} + a_2 x^{2s-2} + \ldots + a_{2s}.$$

On dividing this expression by $x^2 + p_1 x + p_2$, we obtain by the ordinary algebraic rules a remainder of the form $M_{2s-1}x + N_{2s}$, where M_{2s-1} and N_{2s} are functions of p_1 and p_2 whose weights are $2s-1$ and $2s$ respectively, and which may accordingly be written in the forms

$$M_{2s-1} = b_{2s-1} + p_2 b_{2s-3} + \ldots\ldots + p_2^{s-1} b_1,$$
$$N_{2s} = c_{2s} + p_2 c_{2s-2} + \ldots\ldots + p_2^{s},$$

where the b, c are of an order in p_1 indicated by their suffixes. On writing down (by Professor Sylvester's Dialytic method) the result of eliminating p_2 between these equations, it is at once apparent that this resultant is of the order $s(2s-1)$. Thus the determination of a quadratic factor of an expression of degree $2s$ is reduced to the solution of an equation of order $s(2s-1)$. But this number is *one degree more odd* than the original number $2s$; that is to say, if the number $2s$ is 2^k multiplied by an odd number, then $s(2s-1)$ is 2^{k-1} multiplied by an odd number. Hence by a repetition of this process we shall ultimately arrive at an equation of odd order, which, as is well known, must have a real root. By then retracing our steps the existence of a quadratic factor of the original expression is demonstrated.

* [From *Cambridge Philosophical Society's Proceedings*, II. 1876. Read Feb. 21, 1870, pp. 156, 157.]

V.

ON THE SPACE-THEORY OF MATTER*.

(*Abstract.*)

RIEMANN has shewn that as there are different kinds of lines and surfaces, so there are different kinds of space of three dimensions; and that we can only find out by experience to which of these kinds the space in which we live belongs. In particular, the axioms of plane geometry are true within the limits of experiment on the surface of a sheet of paper, and yet we know that the sheet is really covered with a number of small ridges and furrows, upon which (the total curvature not being zero) these axioms are not true. Similarly, he says although the axioms of solid geometry are true within the limits of experiment for finite portions of our space, yet we have no reason to conclude that they are true for very small portions; and if any help can be got thereby for the explanation of physical phenomena, we may have reason to conclude that they are not true for very small portions of space.

I wish here to indicate a manner in which these speculations may be applied to the investigation of physical phenomena. I hold in fact

(1) That small portions of space *are* in fact of a nature analogous to little hills on a surface which is on the average

* [From *Cambridge Philosophical Society's Proceedings*, II. 1876. Read Feb. 21, 1870, pp. 157, 158.]

flat; namely, that the ordinary laws of geometry are not valid in them.

(2) That this property of being curved or distorted is continually being passed on from one portion of space to another after the manner of a wave.

(3) That this variation of the curvature of space is what really happens in that phenomenon which we call the *motion of matter*, whether ponderable or etherial.

(4) That in the physical world nothing else takes place but this variation, subject (possibly) to the law of continuity.

I am endeavouring in a general way to explain the laws of double refraction on this hypothesis, but have not yet arrived at any results sufficiently decisive to be communicated.

[In Paper XV. (infra) reference is made to a *proof that every rational equation has a root*, but I have not found any MS. of which IV. could be looked upon as the Abstract. Prof. Clifford once remarked to me, I think, that a paper on the same subject by Mr J. C. Malet (read before the Mathematical Society, June 14th, 1877, and printed in the *Transactions of the Royal Irish Academy*, Vol. XXVI. No. xiv.) treated the question from a somewhat similar point of view. The subject of V. was introduced at greater length to English mathematicians in a translation (IX. infra) of Riemann's *Habilitationsschrift*, 1854; see the *Gesammelte mathematische Werke*, pp. 254—269.]

VI.

ON JACOBIANS AND POLAR OPPOSITES*.

I. THE word Jacobian is commonly understood to mean a determinant formed from the n^2 differential coefficients of n functions, each of n variables. For instance, given two homogeneous functions of the second degree in x and y, as U, V, each representing two points; the Jacobian is

$$\begin{vmatrix} \frac{dU}{dx}, & \frac{dU}{dy} \\ \frac{dV}{dx}, & \frac{dV}{dy} \end{vmatrix},$$

which is known to represent the foci of the involution determined by U, V. I propose to extend the meaning of the term so as to include *systems* of determinants, formed in precisely the same way from a number of functions *not* equal to the number of variables. For instance, if $U=0$ is the trilinear equation to a conic section, and $L=0$ that to a straight line, then the system of determinants

$$\begin{vmatrix} \frac{dU}{dx}, & \frac{dU}{dy}, & \frac{dU}{dz} \\ \frac{dL}{dx}, & \frac{dL}{dy}, & \frac{dL}{dz} \end{vmatrix} = 0,$$

will be found to represent the pole of the line L with respect to the conic U; and I propose to call this system the Jacobian

* [From *The Oxford, Cambridge and Dublin Messenger of Mathematics*, Vol. II. pp. 229—239.]

of U and L. In all cases I shall denote the Jacobian of a number of functions $A, B, C, \ldots$ whether equal or unequal to the number of variables, by the symbol $J(A, B, C, \ldots)$.

II. As the theory of polar opposites is of constant occurrence in the interpretation of Jacobians, I here set down the heads of it. If the polars of a point A with respect to two conics meet in B, then it is clear that the polars of B will meet in A. On account of this reciprocal relation, I call the points A and B *polar opposites* with respect to the two conics. Consider first one conic; it is known that any line through a point A is harmonically cut by its polar at B and by the conic. The line AB, then, is harmonically divided by both conics, and therefore by any conic through their intersections; *e.g.* by any pair of common chords. It is thus evident that *any* straight line has a pair of polar opposites upon it, which are in fact the foci of the involution in which the line is cut by the conics.

In the same way it may be seen, that if a straight line A joins the poles of B with respect to two conics, B will join the poles of A, and the two lines may be called polar opposites of one another. The point AB is subtended by the two conics and the six intersections of common tangents in an involution of ten rays, of which A and B are the double or sibi-conjugate rays. And through any point we can draw a pair of polar opposite lines, which are in fact the double lines of the involution determined by the four tangents drawn from the given point to the conics.

III. Consider next the case of two conicoids: here the polar planes of any points will intersect on a line, which may be called the polar opposite line of the point. Let A be the point, and BC its opposite line, then it is easy to see that the plane ABC will cut either conicoid in a section, with respect to which A is the pole of BC, and that the same will be true of any conicoid passing through the curve of intersection of two given ones; *e.g.* of the four cones which can be so drawn. This naturally suggests the idea of a triangle, each side of

which is the polar opposite of the corresponding vertex; such a triangle is merely the common self-conjugate triad of the two sections made by its plane. Again, the line BC joining the two poles of any plane A, may be called the polar opposite line of the plane; and the cones drawn from the point ABC touching the two conicoids, will have BC and A for polar line and plane; and the same will be true of the cone drawn touching any conicoid inscribed in the same developable as the two given ones. This again suggests a system of three planes, each the polar opposite of the line of intersection of the other two; but such a system is merely the self-conjugate triad of the two tangent cones drawn to the conicoids from the point of intersection of the planes.

IV. I pass to the case of *three* conicoids. Here a point will have three polar planes, meeting in its polar opposite. Let A and B be opposites, then it is easy to see, as before, that the line AB is harmonically divided by each of the conicoids, and therefore by any conicoid through their eight points of intersection; *e.g.* by any pair of planes which can be drawn to contain the eight points. Similarly, the three poles of a plane A are joined by its polar opposite plane B, and the line AB is subtended by the conicoids in an involution, of which A and B are the double or sibi-conjugate planes.

V. I proceed now to the interpretation of some Jacobians, and commence with those of binary quantics. I shall throughout use the letters L, M, N, ... for equations of the first degree, and generally U, V, ... for those of the second.

1. $J(L, M)$ is obviously the distance between the points L and M*. In general, if L, M are linear functions of any number of variables, $J(L, M) = 0$ is the condition of their coincidence. The distance between two points may also be expressed thus: if Δ stands for $\left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right)$, then ΔL is the

* Strictly, the distance is $\frac{J(L, M)}{(l_1 + m_1)(l_2 + m_2)}$, between $l_1x + m_1y$ and $l_2x + m_2y$.

distance between L and (ξ, η), and generally, with any number of variables, ΔL is proportional to the distance between L and the point $(\xi, \eta, \zeta \ldots)$.

2. $J(U, L)$, where U is a quadric, is the harmonic conjugate of L with respect to U. This may also evidently be expressed by ΔU. For instance, the identity

$$\Delta(\Delta U . \Delta V) = \Delta^2 U . \Delta V + \Delta U . \Delta^2 V$$

may be interpreted thus, if we remember that $\Delta^2 U$ is the product of the distances of (ξ, η) from the two points denoted by U: "Let P and Q be the harmonic conjugates of a point O with respect to A, B and C, D respectively, and let R be the harmonic conjugate of the same point with respect to P, Q; then

$$\frac{RP}{OA . OB} + \frac{RQ}{OC . OD} = 0."$$

3. $J(U, V)$, U and V being quadrics, is well known to represent the foci of the involution determined by U and V.

4. $J(S, L)$, where S is a cubic, is the polar quadric of L with regard to S; that is to say, if S represents the three points A, B, C, then $J(S, L)$ represents two points P, each of which possesses the property

$$\frac{LA}{PA} + \frac{LB}{PB} + \frac{LC}{PC} = 0.$$

They might also evidently have been represented by $\Delta S = 0$. The polar *point*, $\Delta^2 S$, is the harmonic conjugate of L with respect to the two points P; and if we call it R, it possesses the property

$$\frac{RA}{LA} + \frac{RB}{LB} + \frac{RC}{LC} = 0.$$

5. The interpretation of all Jacobians not involving points is easy, when we remember that $J(A, B)$ is in fact the eliminant of ΔA, ΔB. Thus $J(S, U)$, S being a cubic and U a quadric, represents the three points, whose polar points with respect to the cubic, are the same as their harmonic

conjugates with respect to the quadric. If the quadric is the Hessian of the cubic, their Jacobian is no other than the cubicovariant J. (Salmon's *Higher Algebra*, p. 99.) The Hessian represents two points, each of which is the polar quadric of the other; and if we take the harmonic conjugate of each point of the cubic with respect to its Hessian, we shall obtain the covariant J. Let A, B, C be the points of the cubic, and D, E its Hessian; then the three anharmonic ratios

$$[ADBE],\quad [BDCE],\quad [CDAE],$$

are all equal to one another, as appears readily from the canonical form of the cubic. The polar point of any of the points of J with regard to the cubic, being the same as its harmonic conjugate with respect to the Hessian, must be one of the points of the cubic itself; and in this case, the relation at the end of (4), shews that the four points form an harmonic range. We are thus led to Dr Salmon's construction for the covariant J, viz. it contains the harmonic conjugate of each point of the cubic relatively to the other two. In general the Jacobian of a cubic and a quadric does not represent the harmonic conjugates of the cubic with respect to the quadric. $J(S, T)$, S and T being cubics, of course represents the four points whose polar points are the same with regard to the two cubics. The Jacobian of a cubic and its covariant J is the discriminant multiplied by the square of the Hessian.

VI. I put together the most analogous forms of Jacobians in three and four variables, reserving part of the latter for separate consideration.

1. $J(L, M, N) = 0$ and $J(L, M, N, R) = 0$ are known to be respectively the conditions that the three lines and four planes may meet in a point*.

* This Jacobian enables us to express many metrical functions of lines and planes. Let $X=0$ denote the line or plane at infinity, $\phi(L)$ the condition that the line L shall pass through either circular point at infinity, and let

$$\phi(L+\kappa M) \equiv \phi(L) + 2\psi(L, M).\kappa + \phi(M).\kappa^2.$$

Consider now three straight lines L, M, N, and put J for the Jacobian,

2. $J(U, L, M)$, U being a conic, and L, M lines, is the polar of the point LM with respect to U, or the locus of the pole of $lL + mM = 0$. Similarly, if U is a conicoid, and L, M, N planes, $J(U, L, M, N)$ may be interpreted either as the polar plane of the point LMN, or as the locus of the pole of $lL + mM + nN = 0$, l, m, n being arbitrary. $J(U, L)$ means, as before, the pole of L with respect to U.

3. $J(U, V, L)$ is, first, the locus of polar opposites of points on the line L; and, secondly, the locus of the pole of L with respect to $lU + mV = 0$.

If we write U, V in the canonical forms

$$a_1x^2 + b_1y^2 + c_1z^2 = 0,$$
$$a_2x^2 + b_2y^2 + c_2z^2 = 0,$$

then, putting A for the determinant $\begin{vmatrix} b_1, & c_1 \\ b_2, & c_2 \end{vmatrix}$, &c., and

$$lx + my + nz = 0$$

for the equation to the line L, we find the Jacobian to be

$$\frac{lA}{x} + \frac{mB}{y} + \frac{nC}{z} = 0,$$

P for $J(X, M, N) . J(L, X, N) . J(L, M, X)$, and Π for $\phi(L)\ \phi(M)\ \phi(N)$, then

$$\text{area of triangle } LMN \equiv \frac{J^2}{P},$$

$$\text{product of sides} \equiv 2\sqrt{2}\frac{J^3\sqrt{(\Pi)}}{P^2},$$

$$\frac{(\text{area})^2}{\text{product of sides}} \equiv \frac{J}{2\sqrt{(2\Pi)}}$$

(a function of some little importance),

$$\text{radius of circumscribed circle} \equiv \frac{J}{P}\sqrt{(2\Pi)},$$

$$\text{product of sines of angles} \equiv \frac{P}{\Pi}.$$

It is to be noticed that (as is indeed evident) $\phi(L)\ \phi(M) - \psi(L, M)^2$ is identically equal to $J(L, M, X)^2$. See Mr Greer's valuable "Notes," (*Quarterly Journal* for March, 1864).

Similarly, volume of tetrahedron $LENR \equiv \frac{J^3}{P}$, and so on. In all these results there is a factor present, which is easily determined by reference to the fundamental triangle or tetrahedron.

shewing that it always circumscribes the common self-conjugate triangle. The discriminant of this is $ABClmn$, whose evanescence is the condition that L shall pass through one of the vertices of the triangle. Thus we see that *the discriminant with respect to x, y, z of the Jacobian of two conics and the line $\xi x + \eta y + \zeta z = 0$, is a contravariant representing the vertices of the common self-conjugate triangle.* So with four variables, $J(U, V, W, L)$ is both the locus of polar opposites of points in the plane L with regard to U, V, W, and the locus of the poles of L with regard to all the conicoids $lU + mV + nW = 0$. And with a similar notation to that employed above, in case the three conicoids have a common self-conjugate tetrahedron, the Jacobian may be written

$$\frac{lA}{x} + \frac{mB}{y} + \frac{nC}{z} + \frac{rD}{w} = 0,$$

shewing that it contains the edges of the tetrahedron and has double points at the vertices. The discriminant is $ABCDlmnr$, which gives, as before, the theorem, that *when three conicoids have a common self-conjugate tetrahedron, the discriminant with respect to $xyzw$ of the Jacobian of the three conicoids and the line $\xi x + \eta y + \zeta z + \omega w = 0$ is a contravariant representing the vertices of the tetrahedron.* When, in the plane case, the line L is at infinity, the Jacobian represents, as remarked by "Lanivicensis" in the last *Messenger*, the nine-point conic of the quadrangle which is the intersection of U and V; and we see that the polars of any fixed point on this conic, with regard to any conic $lU + mV = 0$, are parallel to a fixed line. In the solid case, it must be remembered that when three conicoids have a common self-conjugate tetrahedron, their eight points of intersection will form a figure which is the "projection" of a parallelepiped, and so $lU + mV + nW$ may in six ways be made to represent two planes. By sending the plane L to infinity, then, we learn that eight points so situated determine a surface of the third order, which bisects the line joining any two of the points, and contains the lines in which intersect the six pairs of planes containing the eight points. Moreover, the section of this surface made by any one of the planes consists of the

line where it meets the corresponding plane, and the nine-point conic of the quadrangle formed on it by the other four planes. And the polar planes of any fixed point on this surface, with respect to all conicoids $lU+mV+nW$, will be parallel to a fixed line.

4. $J(U, V)$ represents the vertices of the common self-conjugate triangle or tetrahedron, according as U and V are conics or conicoids.

5. $J(U, V, L, M)$ is the polar opposite of the point LM with regard to the conics U, V. It is clearly the point common to the Jacobians of U, V, and all lines through the point LM. Also the form of the Jacobian shews that it is common to the polars of LM with respect to all the conics $lU+mV=0$. So, in Solid Geometry, $J(U, V, L, M, N)$ is the polar opposite line of the point LMN; and so also $J(U, V, W, L, M, N)$ is the polar opposite *point* of LMN with respect to U, V, W.

6. $J(U, V, W)$, the Jacobian of three conics, will be found explained at the end of Dr Salmon's *Conic Sections* and in his *Higher Algebra*. I have only to add that it is the locus of the vertices of all triangles self-conjugate to two conics of the forms

$$lU+mV+nW=0, \quad \lambda U+\mu V+\nu W=0,$$

and that lines whose three poles lie in a straight line, meet the lines joining their poles on the Jacobian. So if U, V, W, X are four conicoids, $J(U, V, W, X)$ is the locus of points whose polar planes meet in a point, this latter being also a point on the Jacobian; and the locus of the vertices of all cones which can be represented by

$$lU+mV+nW+rX=0;$$

and consequently the locus of the lines of intersection of all pairs of planes which can be represented by this equation. Moreover it contains the vertices of all tetrahedra self-conjugate with regard to two conicoids of the above form; and the *edges* of all tetrahedra self-conjugate with regard to *three* conicoids of the same form. And obviously, any line whose

polar lines are identical with regard to the four conicoids, will lie on the Jacobian; as is implied by the property last stated.

VII. There are some Jacobians in four variables which have no analogues* in three variables. The simplest is:

1. $J(U, L, M)$, the polar line of LM.

2. $J(U, V, L)$ is the locus of points whose polar opposite lines lie in the plane L, and the locus of the poles of L with regard to all conicoids $lU + mV = 0$. It is a twisted cubic passing through the vertices of the tetrahedron self-conjugate with respect to U and V, and cutting the plane L in the vertices of the self-conjugate triad of the sections in which it meets U, V.

3. $J(U, V, L, M)$ contains the polar opposite lines of all points in the line LM, and the polar lines of LM with regard to all conicoids $lU + mV = 0$. It also contains the poles of all planes of the form $\lambda L + \mu M = 0$ with respect to $lU + mV = 0$. It is clearly a conicoid passing through the vertices of the tetrahedron self-conjugate with regard to U, V, and containing all the twisted cubics

$$J(U, V, lL + mM) = 0.$$

4. $J(U, V, W)$ is a curve of the sixth degree containing the vertices of all cones drawn through the eight intersections of U, V, W (Dr Salmon). Hence it is the locus of the vertices of tetrahedra self-conjugate with regard to two conicoids

$$lU + mV + nW, \quad \lambda U + \mu V + \nu W.$$

If the three conicoids have a common self-conjugate tetrahedron, this sextic represents its edges.

I have given no account of the Jacobians of ternary or quaternary cubics, but their interpretation will be easy with the aid of the following theorem. Consider any number of groups, $A_1, A_2, A_3, \ldots$; $B_1, B_2, \ldots$; $C_1, C_2, \ldots$, &c.; all the

* [1 and 2 seem closely analogous to the 2, 3 of p. 28. C.]

A's being quantics of the same degree, and all the B's, &c., but the A's not necessarily of the same degree as the B's; and form any number of quantics in involution with these, as

$$l_1A_1 + m_1A_2 + n_1A_3 + \ldots\ldots (a_1),$$

$$l_2A_1 + m_2A_2 + n_2A_3 + \ldots\ldots (a_2),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \quad \&c.$$

Now suppose that we have a geometrical interpretation for $J(A, B, C)$, for $J(A_1, A_2, B, C)$, and so on; then

$$J(A_1, A_2, \ldots;\ B_1, B_2, \ldots;\ C_1, C_2, \ldots)$$

will include all the loci formed in the same way as

$$J(a, b, c)^*;\ J(a_1, a_2, \ldots;\ b_1, b_2, \ldots;\ c_1, c_2, \ldots), \text{ \&c.}$$

VIII. I add an example or two to shew that the theory of Polar Opposites is not wholly barren of results.

1. A pair of conjugate diameters cuts the line at infinity in two conjugate points, *i.e.* two points each of which lies in the polar of the other. Hence any two conics have two pairs of conjugate diameters parallel, which cut the line at infinity in a pair of polar opposites†. A particular case is when the axes are parallel, when we see at once, by (II.), that any pair of common chords will be equally inclined to them, and so a circle will pass through the quadrangle of intersection. Similarly, if three conicoids have their principal planes parallel, the eight points of intersection will lie on a sphere.

2. If from the centre O of any circle a normal ON be drawn to a conic, and if two common tangents of the circle and conic intersect in A, and the other two in B, then AN and BN are equally inclined to ON. For it is easily seen that ON and the tangent at N are polar opposite lines. When the point O is on the conic, Professor Cayley has shewn that this

* [? $J(a_1, b_1, c_1)$.]

† [i. e. parallel to each other, and each cutting the line at infinity in one and the same pair of polar opposites. C.]

theorem is a particular case of the theorem that three conics inscribed in the same quadrilateral subtend any vertex in involution (*Educational Times* for December)*. It will be seen that Art. II. is only a development of this remark.

3. The last note on p. 34 of Dr Salmon's *Conics* will be a little plainer if stated as a property of the nine-point conic of any quadrilateral inscribable in a circle. For

$$J(U, V, L^2) \equiv L \,.\, J(U, V, L).$$

So, if U is a sphere, and L the plane at infinity, the twisted cubic $J(U, V, L)$ passes through the feet of the six normals that can be drawn from the centre of U to the conicoid V.

4. If A and B are polar opposites with respect to two circles, AB is bisected by the radical axis, and the circle on AB as diameter cuts both circles orthogonally. Hence, immediately, Dr Salmon's theorem, *Conics*, p. 344, Ex. 3, that the Jacobian of three circles is the circle cutting them orthogonally, together with the line at infinity. It is thus seen that the polars of a point A on the Jacobian meet in the opposite extremity of the diameter through A.

* [*Mathematical Questions with their Solutions*. From the *Educational Times*, Vol. I. p. 33.]

VII.

ON THE PRINCIPAL AXES OF A RIGID BODY*.

THE object of this note is to simplify the manner in which the theory of principal axes is made to depend on the theory of confocal surfaces of the second order.

It is well known†, that if A, B, C are the principal moments of inertia of a rigid body at the centre of gravity, the moment of inertia about an axis through the centre of gravity, whose direction-cosines referred to the principal axes are l, m, n is

$$l^2A + m^2B + n^2C \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

But if M be the mass of the body, and we draw the ellipsoid

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = \frac{1}{M} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

(ellipsoid of gyration), this amounts to saying that the moment of inertia is Mp^2, where p is the perpendicular from the centre on the tangent plane

$$lx + my + nz = p \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

of the ellipsoid (2); that is, if we draw a plane perpendicular to the given axis to touch the ellipsoid (2), then p is the central perpendicular on this plane.

It is also known‡ that the moment of inertia about any

* [From *The Oxford, Cambridge and Dublin Messenger of Mathematics*, Vol. IV. pp. 78—81.]

† Routh's *Rigid Dynamics*, p. 7.

‡ Routh, *l. c.* p. 2.

axis whatever, is equal to the moment about a parallel axis through the centre of gravity, together with the moment which the whole mass, if collected at the centre of gravity, would have about the original axis.

These things being so, the following is a construction for the moment of inertia about any axis PQ through a point P. Draw a plane PT through P perpendicular to the axis; then determine λ so that the surface

$$\frac{x^2}{A+\lambda M}+\frac{y^2}{B+\lambda M}+\frac{z^2}{C+\lambda M}=\frac{1}{M},$$

may touch this plane PT; this gives a simple equation* for λ, which has consequently only one value. Then the required moment of inertia is $M(OP^2-\lambda)$, O being the centre of gravity, which is the origin.

For, draw OT perpendicular to the plane PT; and let Ot be the perpendicular on the parallel plane which touches the ellipsoid of gyration. Then OT being a parallel axis through the centre of gravity, and PT the perpendicular distance of O from PQ; the moment of inertia about PQ is

$$M(Ot^2+PT^2).$$

But† $$OT^2=l^2\left(\frac{A}{M}+\lambda\right)+m^2\left(\frac{B}{M}+\lambda\right)+n^2\left(\frac{C}{M}+\lambda\right)$$

$$=\frac{l^2A+m^2B+n^2C}{M}+\lambda$$

$$=Ot^2+\lambda.$$

Hence $$Ot^2+PT^2=OT^2+PT^2-(OT^2-Ot^2)$$

$$=OP^2-\lambda,$$

the moment of inertia is therefore $M(OP^2-\lambda)$.

Now, consider the confocal ellipsoid which passes through P, and its tangent plane (A) at P. Let λ_1 be the value of λ for this ellipsoid. Then if the plane A be turned ever so little round P, it will begin to cut the ellipsoid in a small ellipse;

* Salmon's *Geometry of three dimensions*, p. 121.

† Salmon, *l. c.* p. 58.

and so the confocal ellipsoid which touches the plane A in its new position, will lie wholly within the other. Its axes will therefore be less than those of the other, and therefore λ will be less than λ_1. That is to say, the value of λ is a *maximum* for the plane which touches the confocal ellipsoid through P. Therefore the moment of inertia about the axis perpendicular to this plane, namely $M(OP^2 - \lambda_1)$, is a *minimum*. Now the axis of least moment at any point is a principal axis. It follows therefore, that *the normal to the confocal ellipsoid through* P *is the principal axis of least moment at* P.

In a manner precisely similar, it may be shewn that if we draw through P an hyperboloid of two sheets confocal to the ellipsoid of gyration, the value of λ for its tangent plane at P is a *minimum;* and therefore that the normal to this surface at P is the principal axis of greatest moment at P.

This being so, we know that the remaining principal axis is perpendicular to these two, and is therefore normal to the confocal hyperboloid of one sheet which passes through P.

We have proved then that the principal axes at any point P, are the normals to the three surfaces confocal to the ellipsoid of gyration which pass through P; and if λ, μ, ν are the values of λ for these three surfaces, or, as we may say, if λ, μ, ν are the curvilinear co-ordinates of P in respect of the ellipsoid of gyration, then the three moments of inertia are

$$M(OP^2 - \lambda), \quad M(OP^2 - \mu), \quad M(OP^2 - \nu).$$

This connection having been established, all the usual propositions about principal axes follow at once from the known theory of confocal surfaces. Thus, the locus of points where two of the principal moments are equal, is the focal conics of the ellipsoid of gyration. For*, of the three confocal surfaces which pass through any point on a focal conic, two coincide with the focal conic itself. For any point on the focal ellipse, the momental ellipsoid is an oblate spheroid; and for

* Salmon, *l. c.* p. 114.

any point on the focal hyperbola, the momental ellipsoid is a prolate spheroid.

Again*, two confocal surfaces can be drawn to touch any given straight line, and the two corresponding tangent planes are at right angles. Hence, on every straight line there are two points at which one principal axis is perpendicular to the line, and these two axes are at right angles. If the two points of contact coincide, then at that point two principal axes are perpendicular to the line, which is therefore itself a principal axis. The condition, therefore, that an axis may be a principal axis at some point of its length, is that the two points of contact of confocal surfaces touching it must coincide; which is obvious, for in that case the line is normal to the third surface passing through the common point of contact.

* Salmon, *l. c.* p. 127.

VIII.

SYNTHETIC PROOF OF MIQUEL'S THEOREM*.

In the note to page 235 of Dr Salmon's *Conics*, mention is made of a theorem originally given by M. Auguste Miquel, in Liouville's *Journal*, Vol. x., p. 349. The theorem may be stated as follows. It is known that we can draw exactly one parabola to touch four given lines. If now we have five lines given, we can draw five parabolas, each of which touches four of the given lines. The theorem is that the foci of these five parabolas lie all on one circle. M. Miquel's proof, reproduced by Catalan, depends on the fact that the circle circumscribing the triangle formed by three tangents to a parabola passes through the focus. Since, as above remarked, a parabola can be drawn to touch any four straight lines, it follows from this that the four circles, circumscribing the four triangles which we get by leaving out each of the lines in turn, all meet in a point, the focus of the parabola. The theorem can thus be stated as a property of straight lines and circles, without any mention of parabolas; and accordingly M. Miquel proves it by using the ordinary (Euclid) geometry appropriate to such theorems.

I find, however, that the theorem connects itself in a somewhat interesting way with that general conception of geometrical facts which regards all the properties of a curve or other continuous figure as depending upon its order alone, and which it has become usual to call Synthetic Geometry. I am not

* [From *The Oxford, Cambridge and Dublin Messenger of Mathematics*, Vol. v. pp. 124—141.]

satisfied with the word, and use it under protest for want of a better. The *thing* should be more widely known than it is in this country; the simplicity and instructiveness of its application in the present case are my reasons for calling attention to a matter otherwise unimportant. Those who are of the mind of Herr Gretschel may substitute for the words "Synthetic Proof," "Proof by *Organic* Geometry."

I.

On the Shapes of Certain Curves.

It is very important that we should know what we mean by a *curve*. To this end I start with Plücker's mode of generating curves.

Imagine a point and a straight line passing through it to be moved about on a plane in this manner. The point always moves along the line, never stopping for any portion of time however small; and all this while the line is turning round the point and never stops for any portion of time however small. Then the point will *describe*, and the line will *envelope*, a curve. To get a clear notion of this, take the particular case of a line being rolled round a circle (without sliding) so as always to touch it. The point of contact will then be always moving along the tangent, and the tangent will always be turning round the point of contact. But, at the same time, the point will by its motion have traced out the circle; and if we imagine all that part of the plane which the line passes over to become black, there will be left a white patch bounded by this circle, *enveloped* by all the positions of the moving line.

Now here there are three things to consider:—

A. The actual curve which you see, dividing that portion of the plane which is inside it from the portion which is outside it.

B. The assemblage of all the positions of the moving point.

C. The assemblage of all the positions of the moving line.

It is very easy but at the same time very important to observe that these are three distinct things. We are accustomed to say that A is the *locus* of B (the points) and the *envelope* of C (the tangents). Every curve of course has an assemblage of points upon it, and an assemblage of lines touching it; but it is not *the same thing* as either of these, any more than the assemblage of points is the same thing as the assemblage of lines. I shall illustrate this further by taking the two simplest and most fundamental cases.

Namely, suppose first that the point, instead of moving, remains always at rest while the line turns round it. Then the figure A is merely the point itself and no longer a curve. B has entirely disappeared; there is no assemblage of positions of the moving point. For by an *assemblage* of positions [of a point] we mean at least a line; now a point is absolutely *no* line, as a line is *no* surface, and a surface *no* space. C is now the assemblage of lines through the point. So then of our three things A and C remain, while B has entirely disappeared; but it is exceedingly obvious that a point and the assemblage of straight lines through it are two different things. Next suppose that the line remains still while the point moves along it. Then A is the straight line, which we are not accustomed to call a curve. B is the assemblage of all the points on this line. C has entirely disappeared, for there is no assemblage of lines. But a straight line is not the same thing as all the points on it, though you may think so at first. To be convinced, contemplate the other case just considered; you have just as much right to say that a point is the same thing as all the lines through it. And then read S. Thomas Aquinas on this question, if you can find the reference, which I have forgotten.

An assemblage of points is said to be of a certain *order*, when a certain number of the points can be found upon an arbitrary straight line. Thus the assemblage of points lying upon a straight line is of the *first* order, because *one* of the points can be found upon another arbitrary straight line.

The assemblage of points lying upon two straight lines is of the second order, since two of them can be found upon another arbitrary straight line. And generally the assemblage of all the points lying upon n straight lines is of the nth order.

Similarly, an assemblage of lines is said to be of a certain *class*, when a certain number of the lines can be drawn through an arbitrary point. Thus the assemblage of lines passing through a point is of the *first* class, because *one* of the lines can be drawn through another arbitrary point. The assemblage of lines passing through two points is of the second class, because two of the lines can be drawn through another arbitrary point. And generally the assemblage of all the lines passing through n points is of the nth class.

We have now a number appertaining to the aggregate of points, viz. its *order*, and a number appertaining to the aggregate of lines, viz. its *class*. Neither of these numbers belong in strictness to the curve itself; but there is a number—the *Geschlecht-zahl* or *deficiency*—which does belong to the curve, and not immediately to the points or tangents*. It is not however my business to speak of that now. I am going to investigate certain cases in which these two things—the assemblage of points, and the assemblage of tangents—change continuously together; and in which it is very important to observe the modifications which *both* of them undergo.

A curve of the third class is, in general, of the shape represented in fig. 1; that is to say, it consists of a tricusp surrounded by an oval. It is very easy to see that from any point within the tricusp we can draw three tangents to it and none to the oval; from any point in the intermediate space we can draw one tangent to the tricusp and none to the oval; and from any point outside the oval we can draw two tangents to it and one to the tricusp. Such a curve is of the sixth order; and it is singled out among curves of the

* [This incidental remark seems noteworthy: the number in question belongs as much, and in the same way, to the tangents as to the points, that is, not peculiarly to either: but the author's point of view seems to be a different one. C.]

sixth order as having nine cusps, of which as you see only three are real. The tangents at three real cusps always meet in a point*.

Here then is an assemblage of points which is of the sixth order connected with an assemblage of lines which is of the third class by the fact that they are respectively the points and tangents of a certain curve. I am going to alter the curve, and to watch what becomes of these two assemblages.

Imagine now that two of the cusps approach very near to the oval, as in fig. 2. I call these two cusps *a* and *b*, and I want to attend particularly to two portions of the curve; namely, (1) the branch of the tricusp which is between these two cusps, and (2) the portion of the oval which is between the two points to which the cusps are very near. Suppose these two portions to become flatter and flatter, and to approach nearer and nearer to each other; what becomes of the tangents to them? All these tangents get to differ less and less from the line joining the two cusps. At last, suppose that they all coincide with it, and let us watch what becomes in this case of the curve, its points, and its tangents. First, for the curve; the two other branches of the tricusp join on to the remaining portion of the oval and form a figure like a cardioid, represented in fig. 3. The assemblage of tangents must remain of the third class; it consists just of the tangents to this cardioidal curve, *among which the line ab counts for two.* Lastly, for the points; these are in the first place the points of the cardioidal curve afore-mentioned, clearly enough. But besides these we have to account for the points on the two portions which coincided with the segment *ab.* These obviously pass continuously into the points upon the linear segment *ab*, and each of these counts for two. Moreover, there were two series of invisible points of the curve very near to all the rest of the line *ab*, outside this segment; and at the instant that the two visible portions united on the segment, these invisible portions started into

* Mr Cotterill is, I believe, the first person that ever *saw* a curve of the third class.

visible existence by uniting all along the rest of the straight line. I affirm this dogmatically, and there is no reason for you to believe it, unless you are acquainted with the theory of invisible points. Our assemblage of points which was of the sixth order has then broken up into the points on a cardioidal curve of the fourth order, and the points on a straight line each counting for two.

Other changes have taken place at the same time. I said that the original curve of the third class and sixth order had nine cusps, three visible and six invisible. In the transformation which the curve has undergone, we have seen it acquire a double tangent and lose two visible cusps. Now I assert—dogmatically as before—that besides these two visible cusps it has also lost four of the invisible ones; so that the cardioidal curve which is left has three cusps, one visible and two invisible. And it is a general rule (discovered by Plücker and explained in Dr Salmon's *Higher Plane Curves*, p. 73) that every curve of given class which acquires a double tangent loses thereby six cusps; exactly as a curve of given order which acquires a node loses thereby six inflexions. Of course it is now natural to ask "can we not give the curve a double tangent in such a way as to cut off all the invisible cusps, and leave only the three real ones?" We can do this; but it is necessary first to enter into explanations in regard to two possible kinds of double tangents. Precisely as a double point may be either a point at which two visible branches of a curve cross, or else a conjugate point, the limit of a very small oval; so a double tangent may either have two visible points of contact (as in fig. 3), or two invisible ones*. In the latter case it is called an *ideal* tangent (Poncelet) and appears to have—like a conjugate point—nothing to do with the curve. Now a curve of the third order (fig. 4) may have a double point given to it in two ways. Either we may visibly join together the oval and the sinuous part, making a loop (fig. 5); or we may let the oval shrink up into a conjugate point, at which two invisible

* Salmon's *Higher Plane Curves*, pp. 34, 35. I learn with great satisfaction that already the new edition of this is partly in print. [Published Jan., 1873. The work is now, August, 1879, in a third edition.]

branches of the curve cross each other (fig. 6). Precisely analogous distinctions hold between the two ways in which a curve of the third class (fig. 7) may acquire a double tangent. (Fig. 7 is the same curve as fig. 1, but it has been projected so that the line at infinity cuts the oval part, which consequently resembles a hyperbola instead of an ellipse.) We may either make this go through the process already described, by which it becomes fig. 8, acquiring a real double tangent with visible points of contact; or we may make the angle between the asymptotes larger and larger, till the two branches of the hyperbolic part coincide into a doubled straight line (fig. 9), which is then an ideal double tangent having invisible points of contact. And this kind of double tangent does what was wanted; viz. it removes six invisible cusps, leaving three visible ones.

We establish then two different kinds of curves of the third class having a double tangent. First, there is the cardioidal curve, having visible points of contact with its double tangent, one visible and two invisible cusps. Secondly, there is the simple tricusp, having invisible points of contact with a real double tangent, and three visible cusps. My main business is with the first of these kinds; but before coming to it, I shall make some remarks about two special forms, one of each kind, and mention also some apparently different general forms of curves of the third class not having a double tangent.

Of the cardioidal form the cardioid itself is a particular case; viz. it is a curve of the third class and fourth order with one visible and two invisible cusps. In general, a curve of the fourth order which has cusps at the two invisible points at infinity through which all circles pass is called a Cartesian oval. A Cartesian oval may also be defined as the locus of a point whose distances (ρ, ρ') from two fixed points satisfy an equation of the form $m\rho + n\rho' = c$. The curve has three foci, all in the same straight line, and any two of them may be taken for the fixed points. A Cartesian oval with an additional cusp (this is necessarily real) is a cardioid; the three foci coincide at the cusp. Any curve of our first kind

then may be regarded as the shadow of a cardioid; and in order to project such a curve into a cardioid, it is only necessary to project the two invisible cusps into the circular points at infinity*.

A particular case of the other form is the hypocycloid of three branches. This curve has for its double tangent the line at infinity, and touches it at the circular points. Every curve of the third class with an ideal double tangent may thus be regarded as the shadow of a hypocycloid; and in order to project it into a hypocycloid, it is only necessary to project the invisible points of contact into the circular points at infinity.

This curve is the envelope of the asymptotes of all the rectangular hyperbolas that pass through three fixed points. (Steiner). To prove this, it is necessary to shew first that the envelope is of the third class, *i.e.* that three such asymptotes can be drawn through an arbitrary point; and secondly, that the line infinity counts twice as an asymptote, or is a double tangent to the envelope, its points of contact being the two circular points. Now a rectangular hyperbola is one which cuts the line infinity in two points which make with the circular points a harmonic range. If the hyperbola *touch* the line at infinity (become a parabola), its two points of intersection coincide; and they can only do this at one or other of the circular points. There are therefore these two cases in which the line infinity is itself an asymptote; and if we consider a hyperbola very near to one of these cases, we see that the point at which the line infinity is met by the consecutive asymptote is the circular point itself. We have therefore established the *second* of our two facts; that the line infinity is a double tangent to the envelope, [and that the] points of contact are the circular points. To find now the *class* of the envelope, let us enquire how many tangents can be drawn to it through an

* On Cartesian Ovals see Crofton, *Proceedings of the London Mathematical Society* [Vol. VI. pp. 5—18]. On the Cardioid, Purkiss, *Messenger of Mathematics*, [Vol. II. pp. 241—249], and in especial relation to the present theory, Siebeck, *Ueber die Erzeugung der Curven* 3ter *Klasse und* 4ter *Ordnung durch Bewegung eines Punktes*, Crelle, Vol. LXVI. p. 344 (1866).

arbitrary point at infinity. There is in the first place the line infinity itself, counting for two. Besides this, there is the asymptote of the *one* hyperbola of the series that can be drawn through the given point; in all *three* tangents, or the envelope is of the third class. But every curve of the third class touching the line infinity at the circular points is a three-cusped hypocycloid: therefore, &c.*

I said that *in general* a curve of the third class consists of a tricusp surrounded by an oval. But an oval is a thing of the nature of a conic section, and may at times be wholly invisible, like the conic section $x^2 + y^2 + a^2 = 0$. We have accordingly a variety in which the oval has disappeared, and the curve is represented by a tricusp only. The tricusp, however, need not be finite; I have represented in fig. 10 a curve met by every straight line in at least two real points, which cannot therefore be projected into any finite form. To satisfy yourself that it is really a tricusp very similar to fig. 9, draw on a sphere its curve of intersection with a cone, whose vertex is the centre, standing on this curve; it will consist of two equal and opposite tricusps, each with two branches longer than half a great circle.

II.

The Focus of a Double Parabola.

I shall take the liberty of using the name *Double Parabola* to denote a curve of the third class, having the line infinity for a double tangent; the points of contact being any two points whatever at infinity. The reason of the name is tolerably obvious; for the curve has two pair of parabolic branches, and may be derived from the *ensemble* of two parabolas by continuous modification. If the two points of contact are visible, the curve is a central projection of a car-

* On the three-cusped hypocycloid, see a series of papers in the *Educational Times*, Reprint, Vols. III., IV.; and a most elegant synthetic discussion by Cremona, *Sur l'hypocycloide à trois rebroussements*, which appeared about the same time in Crelle, Vol. LXIV. p. 101 (1865).

dioid, obtained by projecting the double tangent to infinity; if invisible, the curve is an orthogonal projection of a hypocycloid. The two kinds may be distinguished as *hyperbolic* and *elliptic* respectively—names on which more light will be thrown in the sequel. The hypocycloid of three branches must then be regarded, in accordance with this nomenclature, as a *circular* double parabola.

Now a *focus* of a curve is a point such that the two lines joining it to the circular points at infinity both touch the curve. Accordingly, a double parabola has in general one focus and no more than one; for the curve is of the third class, and consequently only one tangent can be drawn from one of the circular points, besides the line infinity which counts for two. This single focus lies inside the hyperbolic curve, and outside the elliptic one, moving further away from the curve as it approaches the circular form, the focus of which is away at infinity in no particular direction.

I am going to seek for a geometrical property of this focus, which may serve as a rule for constructing the curve. To this end I form as follows the tangential equation of the curve. Let $f=0$ be the tangential equation of the focus, $i=0$, $j=0$, the equations of the circular points at infinity; $a=0$, $b=0$ the equations of the points of contact with the line at infinity, and $c=0$ the point at infinity on the remaining tangent from the focus to the curve. Then I say that the equation is

$$abf = \lambda . ijc \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where λ is a numerical constant. For the equation is of the third degree, representing therefore a curve of the third class; it is satisfied by the coordinates of the lines fi, fj, fc, shewing that fi, fj are tangents to the curve, and f consequently a focus, and that fc is the remaining focal tangent; and it shews that the three tangents from each of the points a, b (viz. ai aj, ac; bi, bj, bc) coincide with the line at infinity—the points a, b, c, i, j being all on that line—so that a and b must be points on the curve at which that line touches it, unless the line were a triple tangent, which is impossible for a curve of the third class.

To make this equation yield us a relation of lengths and angles serving for geometrical construction, let us take lines A, B, C at a finite distance passing through the points a, b, c respectively, and denote by the letter O the line at infinity. The points a, b, c may now be denoted by AO, BO, CO; viz. a is the intersection of the lines A, O and so on. Let X be a variable tangent to the curve; then substituting its coordinates in the equation (1) thus modified, we have

$$XAO \,.\, XBO \,.\, Xf = \lambda \,.\, Xi \,.\, Xj \,.\, XCO \ldots\ldots\ldots (2),$$

where XAO means the determinant formed with the coordinates of the lines X, A, O, which vanishes when they meet in a point; and Xf means the result of substituting the coordinates of the line X in the equation of the point f, or (which is the same thing) the result of substituting the coordinates of the point f in the equation of the line X. Now if we observe that*

$$\sin(X, A) = \frac{XAO}{\sqrt{(Xi \,.\, Xj \,.\, Ai \,.\, Aj)}},$$

$$\text{perpendicular from } f \text{ on } X = \frac{Xf}{\sqrt{(Xi \,.\, Xj) \,.\, Of}},$$

* These formulæ are justified as follows. The coordinates of the circular points are taken to be

$$\text{of the point } i,\ x : y : 1 = 1 : \iota : 0,$$
$$\text{of the point } j,\ x : y : 1 = 1 : -\iota : 0,$$

and the equation of the line O at infinity is taken to be

$$0 \,.\, x + 0 \,.\, y + 1 = 0.$$

Then if the equation of X is $lx + my + n = 0$, we have

$$Xi \,.\, Xj = (l + m\iota)(l - m\iota) = l^2 + m^2,$$

and if the equation of A is $l'x + m'y + n' = 0$, then

$$XAO = \begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ 0, & 0, & 1 \end{vmatrix} = lm' - l'm,$$

while $Of = 1$, whatever are the coordinates of f. Making these substitutions, the formulæ become

$$\sin(X, A) = \frac{lm' - l'm}{\sqrt{(l^2 + m^2)}\sqrt{(l'^2 + m'^2)}},$$

$$\text{perpendicular from } f \text{ on } X = \frac{lx' + my' + n}{\sqrt{(l^2 + m^2)}} \ (x'y' \text{ coordinates of } f),$$

which are the ordinary ones.

we may transform equation (2) into the following

$$\frac{\text{perpendicular from } f \text{ on } X}{\sin(X, C)} = \frac{\mu}{\sin(X, A)\sin(X, B)} \quad \ldots\ldots (3),$$

where $$\mu = \frac{\lambda \sqrt{(Ci \,.\, Cj)}}{\sqrt{(Ai \,.\, Aj \,.\, Bi \,.\, Bj)} \,.\, Of},$$

Now C was any line through the point c, that is, any line parallel to the focal tangent; but I shall now regard it as the focal tangent itself. This being so, it is clear that $\frac{\text{perpendicular from } f \text{ on } X}{\sin(X, C)}$ = distance from f to X, measured along the focal tangent; or it is the segment on the focal tangent determined by the variable tangent X. We have arrived then at the theorem that *this distance is inversely proportional to the product of the sines of the angles which the tangent makes with the asymptotic directions* A *and* B. Let δ be the distance from the focus to the point of contact of the focal tangent; then taking X to be the focal tangent itself, we get

$$\delta = \frac{\mu}{\sin(A, C)\sin(B, C)},$$

and, eliminating μ, we may state our theorem in the form

$$\frac{\text{segment on focal tangent}}{\delta} = \frac{\sin(A, C)\sin(B, C)}{\sin(A, X)\sin(B, X)},$$

= anharmonic ratio of points a, b, c, XO.

We may simplify still further this enunciation by remembering that any squared diameter of a conic is inversely proportional to the product of the sines of the angles which it makes with the asymptotes, and at the same time directly proportional to the parallel focal chord. Finally, we arrive at the following

Construction. On any fixed line C through a focus f of a conic section let a distance fp be measured equal to a focal chord, and through p let a line X be drawn parallel to the focal chord; the envelope of the line X as the chord turns round will be a double parabola which has infinite branches

parallel to the asymptotes of the conic, the point f for focus, and the line C for focal tangent.

According as the conic is ellipse or hyperbola, the double parabola will be elliptic or hyperbolic. If the conic is a parabola, it assumes an intermediate form, the semi-cubical parabola $ay^2 = x^3$. If the conic is a circle, the envelope reduces to a point, as it ought to, being a hypocycloid whose focus is at an infinite distance compared with the dimensions of the figure.

III.

THEOREM. *The locus of the foci of all the double parabolas which touch five fixed lines is a circle.*

A double parabola is uniquely determined by six tangents. For a curve of the third class is determined in general by nine tangents; and to be given that a particular line is a double tangent is equivalent to three linear tangential conditions. The double parabolas touching five fixed lines are therefore a singly infinite series like the conics inscribed in a quadrilateral, and there is only one of them that touches another arbitrary line. Let an arbitrary line be drawn through the circular point i; there is then one double parabola of the series that touches this line. From the point j one tangent only can be drawn to the curve, for it is of the third class, and the line infinity already counts for two tangents. When therefore the tangent from i to the curve is given, the tangent from j is uniquely determined; and so likewise when the tangent from j is given, the tangent from i is uniquely determined. These two tangents are therefore corresponding rays of two homographic pencils, and the locus of their intersection (*i.e.* of the focus f) is consequently a conic section passing through the points i, j, that is to say, a circle.

COR. 1. The foci of the five parabolas which touch every four of the five lines are on this circle.

For a double parabola, being a curve of the third class, may break up into a conic and a point; namely, into an ordinary parabola and a point at infinity. Among the double parabolas which touch the five lines are to be reckoned five such degenerate cases, consisting of the point at infinity on one of the lines and the conic parabola which touches the other four. The focus of this degenerate form is (as is obvious from the definition of a focus) simply the focus of the conic parabola; whence the corollary follows, and Miquel's theorem is proved.

COR. 2. If we take six lines to start with, we may in this way determine six circles, omitting the lines one by one. These six circles all meet in a point, the focus of the double parabola which touches the six lines.

The transformation (as to its tangents) of a cardioidal curve into a conic and a point is illustrated by fig. 11, which represents such a curve very nearly consisting (as to its visible points) of a conic and a doubled finite portion of one of its tangents.

IV.

DEVELOPMENTS.

So far we have considered the following series of propositions:

(1). Given three lines, a circle may be drawn through their intersections. (Euc. IV. 5.)

(1′). Given four lines, the four circles so determined meet in a point. (Well known.)

(2). Given five lines, the five points so found lie on a circle. (Miquel.)

(2′). Given six lines, the six circles so determined meet in a point. (Sect. III., Cor. 2.)

I shall now shew that the series is interminable; that is, that $2n$ lines determine $2n$ circles all meeting in a point, and

that for $2n+1$ lines the $2n+1$ points so found lie on the same circle.

Connected with Prop. 1, however, there is another theorem which is susceptible of generalization. If from any point in the circumscribing circle we draw perpendiculars to the sides of a triangle, the feet of these perpendiculars are in one straight line. I call this $(1p)$; the corresponding pendant to Miquel's theorem is,

$(2p)$. If from any point p on the circle in Prop. (2) we draw perpendiculars on the five lines, their feet lie on a conic passing through p.

So again we have

(3). Given seven lines, the seven points obtained as in (2′), are all in one circle.

$(3p)$. If from any point p on this circle we draw perpendiculars on the seven lines, their feet will lie on a cubic having a node at p.

And generally

(np). If from any point p on the circle determined by $2n+1$ lines we draw perpendiculars to them, the feet of the perpendiculars will lie on a curve of order n passing $n-1$ times through p.

To prove these results, it is necessary to consider a curve of class $n+1$, touching the line infinity n times. I shall call such a curve an n-fold parabola. It is in fact of order $2n$, and has n pairs of parabolic branches*. From any point at infinity

* The tangential equation to an n-fold parabola is always

$$fa_1a_2 \dots a_n = \lambda . ijc_1c_2 \dots c_{n-1},$$

where a_1, a_2, ... are points of contact with the line infinity, and c_1, c_2, ... points at infinity on tangents from the focus to the curve. From this we may deduce at once a construction analogous to (3), p. 49. In the case of the triple parabola this construction takes the simplified form

$$\text{intercept on } X \text{ made by } A_1,\ A_2 = \frac{\mu}{\sin(X, C_1)\sin(X, C_2)\sin(X, C_3)},$$

where A_1, A_2 are the focal tangents, and C_1, C_2, C_3 the asymptotic directions. μ may be determined by making X coincide with A_1, A_2 successively.

not a point of contact, there may be drawn one other tangent to the curve; the line infinity counting for n, and the class being $n+1$. Such a curve therefore has always one and only one focus. Now a curve of class $n+1$ is in general determined by $\frac{1}{2}(n+2)(n+3)-1$ or $\frac{1}{2}(n^2+5n+4)$ tangents; but an n-fold tangent of given position is equivalent to $\frac{1}{2}n(n+1)$ single tangents in its effect upon the determination of the curve. The number of tangents finite in position which determine an n-fold parabola is the difference of these numbers, $2(n+1)$. All the n-fold parabolas, therefore, which touch $2n+1$ fixed lines form a singly infinite series; and it is easy to see that the locus of their foci is a circle. For if we draw an arbitrary line through the circular point i, *one* curve of the series can be drawn to touch it, and this determines uniquely the tangent from the other point j. These two tangents then, as before, are corresponding rays of two homographic pencils, and their intersection must trace out a conic through the points i, j, that is to say, a circle.

Now among these n-fold parabolas are included $2n+1$ degenerate cases, each consisting of an $(n-1)$ fold parabola and a point, viz. the point at infinity on one of the lines, and the $(n-1)$ fold parabola determined by the other $2n$. The foci of these are therefore points on the circle in question, and we may enunciate the following propositions:

(n). Given $2n+1$ lines, the foci of the $2n+1$ $(n-1)$ fold parabolas each of which touches $2n$ of the lines are on the same circle.

This circle is the locus of the foci of n-fold parabolas touching the lines, and therefore

(n'). Given $2n+2$ lines, the $2n+2$ circles so determined meet in a point, the focus of the n-fold parabola touching the lines.

By successive applications of this theorem with suitable values of n, we are able to see that the series of statements at the beginning of this section may be continued indefinitely.

Let us now inquire what is the *pedal* of an n-fold parabola with regard to the focus, that is to say, what is the locus of the

foot of the perpendicular from the focus on the tangent. It is easy to prove—as was pointed out by Dr Hirst—that this pedal is the inverse of the polar reciprocal of the curve, both taken with regard to a circle whose centre is the focus. Now the reciprocal of an n-fold parabola in regard to the focus is a curve of the $(n+1)^{\text{th}}$ order passing n times through the focus (because the original curve touches n times the line infinity) and once through each of the circular points (because the original curve touches the lines fi, fj). Its inverse is then a curve of order $2(n+1)-n-2=n$, passing $n+1-2=n-1$ times through f, and not at all through the circular points*. Hence we have the proposition

(*np*). From any point which is the focus of an n-fold parabola touching $2n+1$ lines, if perpendiculars be drawn to the lines, their feet will lie on a curve of order n passing $n-1$ times through the point in question.

* Circular inversion (or transformation by reciprocal radii-vectores) is a particular case of triangular or quadric inversion; the pole and the two circular points forming the triangle employed. Now in general the triangular inverse of a curve of order n, passing α, β, γ times respectively through the vertices of the triangle, is a curve of order $2n-\alpha-\beta-\gamma$ passing $n-\beta-\gamma$, $n-\gamma-\alpha$, $n-\alpha-\beta$ times respectively through the vertices of the triangle. (Dr Hirst, *Proceedings of the Royal Society*, March, 1865; and *Educational Times*, Reprint, Vol. I. p. 41; Vol. III. p. 91.)

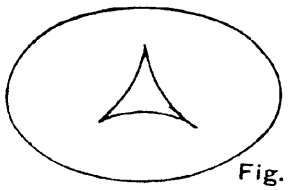


Fig.1.

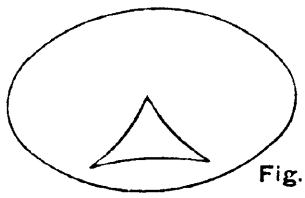


Fig.2.

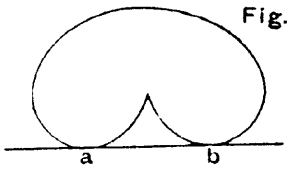


Fig.3.

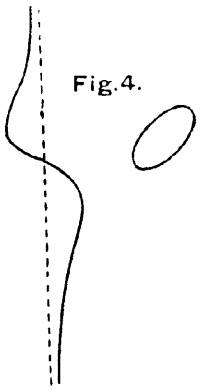


Fig.4.

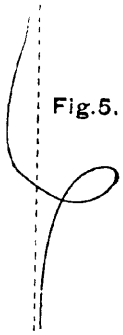


Fig.5.

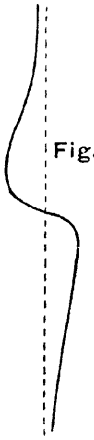


Fig.6

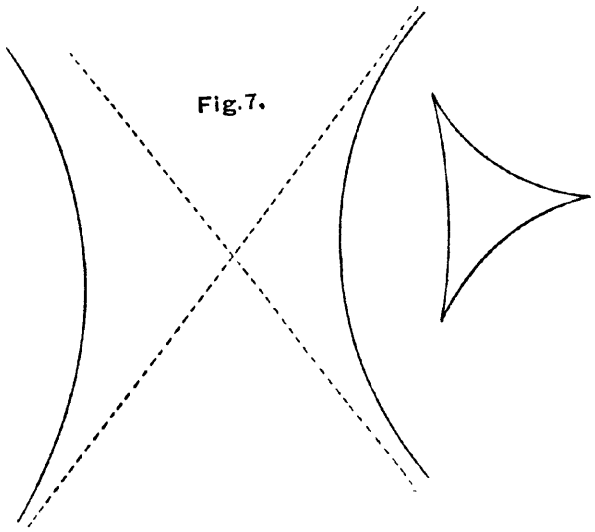


Fig.7.

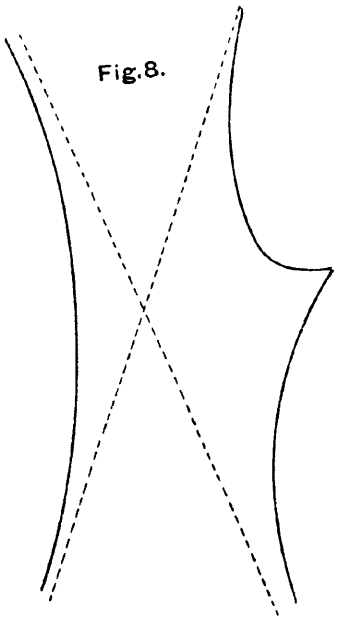


Fig.8.

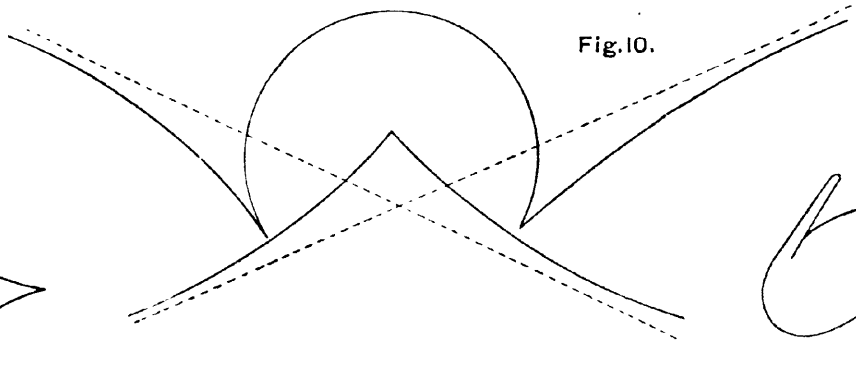


Fig.10.

Fig.9.

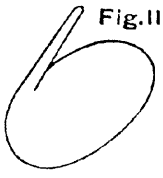


Fig.11.

face p. 54.

IX.

ON THE HYPOTHESES WHICH LIE AT THE BASES OF GEOMETRY*.

[Translation of a paper by Riemann, see V. supra.]

Plan of the Investigation.

It is known that geometry assumes, as things given, both the notion of space and the first principles of constructions in space. She gives definitions of them which are merely nominal, while the true determinations appear in the form of axioms. The relation of these assumptions remains consequently in darkness; we neither perceive whether and how far their connection is necessary, nor, *a priori*, whether it is possible.

From Euclid to Legendre (to name the most famous of modern reforming geometers) this darkness was cleared up neither by mathematicians nor by such philosophers as concerned themselves with it. The reason of this is doubtless that the general notion of multiply extended magnitudes (in which space-magnitudes are included) remained entirely unworked. I have in the first place, therefore, set myself the task of constructing the notion of a multiply extended magni-

* [From *Nature*, Vol. VIII. Nos. 183, 184, pp. 14—17, 36, 37. For a Bibliography of Hyper-space and Non-Euclidean Geometry, see Articles by George Bruce Halsted in the *American Journal of Mathematics, Pure and Applied*, Vol. I. pp. 261—276, 384, 385; Vol. II. pp. 65—70.]

tude out of general notions of magnitude. It will follow from this that a multiply extended magnitude is capable of different measure-relations, and consequently that space is only a particular case of a triply extended magnitude. But hence flows as a necessary consequence that the propositions of geometry cannot be derived from general notions of magnitude, but that the properties which distinguish space from other conceivable triply extended magnitudes are only to be deduced from experience. Thus arises the problem, to discover the simplest matters of fact from which the measure-relations of space may be determined; a problem which from the nature of the case is not completely determinate, since there may be several systems of matters of fact which suffice to determine the measure-relations of space—the most important system for our present purpose being that which Euclid has laid down as a foundation. These matters of fact are—like all matters of fact—not necessary, but only of empirical certainty; they are hypotheses. We may therefore investigate their probability, which within the limits of observation is of course very great, and inquire about the justice of their extension beyond the limits of observation, on the side both of the infinitely great and of the infinitely small.

I. *Notion of an* n*-ply extended magnitude.*

In proceeding to attempt the solution of the first of these problems, the development of the notion of a multiply extended magnitude, I think I may the more claim indulgent criticism in that I am not practised in such undertakings of a philosophical nature where the difficulty lies more in the notions themselves than in the construction; and that besides some very short hints on the matter given by Privy Councillor Gauss in his second memoir on Biquadratic Residues, in the *Göttingen Gelehrte Anzeige*, and in his Jubilee-book, and some philosophical researches of Herbart, I could make use of no previous labours.

§ 1. Magnitude-notions are only possible where there is an antecedent general notion which admits of different specialisa-

tions. According as there exists among these specialisations a continuous path from one to another or not, they form a *continuous* or *discrete* manifoldness: the individual specialisations are called in the first case points, in the second case elements, of the manifoldness. Notions whose specialisations form a *discrete* manifoldness are so common that at least in the cultivated languages any things being given it is always possible to find a notion in which they are included. (Hence mathematicians might unhesitatingly found the theory of discrete magnitudes upon the postulate that certain given things are to be regarded as equivalent.) On the other hand, so few and far between are the occasions for forming notions whose specialisations make up a *continuous* manifoldness, that the only simple notions whose specialisations form a multiply extended manifoldness are the positions of perceived objects and colours. More frequent occasions for the creation and development of these notions occur first in the higher mathematic.

Definite portions of a manifoldness, distinguished by a mark or by a boundary, are called Quanta. Their comparison with regard to quantity is accomplished in the case of discrete magnitudes by counting, in the case of continuous magnitudes by measuring. Measure consists in the superposition of the magnitudes to be compared; it therefore requires a means of using one magnitude as the standard for another. In the absence of this, two magnitudes can only be compared when one is a part of the other; in which case also we can only determine the more or less and not the how much. The researches which can in this case be instituted about them form a general division of the science of magnitude in which magnitudes are regarded not as existing independently of position and not as expressible in terms of a unit, but as regions in a manifoldness. Such researches have become a necessity for many parts of mathematics, e.g., for the treatment of many-valued analytical functions; and the want of them is no doubt a chief cause why the celebrated theorem of Abel and the achievements of Lagrange, Pfaff, Jacobi for the general theory of differential equations, have so long remained unfruitful. Out of this general part of the science of extended magnitude in which nothing is assumed

but what is contained in the notion of it, it will suffice for the present purpose to bring into prominence two points; the first of which relates to the construction of the notion of a multiply extended manifoldness, the second relates to the reduction of determinations of place in a given manifoldness to determinations of quantity, and will make clear the true character of an n-fold extent.

§ 2. If in the case of a notion whose specialisations form a continuous manifoldness, one passes from a certain specialisation in a definite way to another, the specialisations passed over form a simply extended manifoldness, whose true character is that in it a continuous progress from a point is possible only on two sides, forwards or backwards. If one now supposes that this manifoldness in its turn passes over into another entirely different, and again in a definite way, namely so that each point passes over into a definite point of the other, then all the specialisations so obtained form a doubly extended manifoldness. In a similar manner one obtains a triply extended manifoldness, if one imagines a doubly extended one passing over in a definite way to another entirely different; and it is easy to see how this construction may be continued. If one regards the variable object instead of the determinable notion of it, this construction may be described as a composition of a variability of $n+1$ dimensions out of a variability of n dimensions and a variability of one dimension.

§ 3. I shall now show how conversely one may resolve a variability whose region is given into a variability of one dimension and a variability of fewer dimensions. To this end let us suppose a variable piece of a manifoldness of one dimension—reckoned from a fixed origin, that the values of it may be comparable with one another—which has for every point of the given manifoldness a definite value, varying continuously with the point; or, in other words, let us take a continuous function of position within the given manifoldness, which, moreover, is not constant throughout any part of that manifoldness. Every system of points where the function has a constant value, forms

then a continuous manifoldness of fewer dimensions than the given one. These manifoldnesses pass over continuously into one another as the function changes; we may therefore assume that out of one of them the others proceed, and speaking generally this may occur in such a way that each point passes over into a definite point of the other; the cases of exception (the study of which is important) may here be left unconsidered. Hereby the determination of position in the given manifoldness is reduced to a determination of quantity and to a determination of position in a manifoldness of less dimensions. It is now easy to show that this manifoldness has $n-1$ dimensions when the given manifoldness is n-ply extended. By repeating then this operation n times, the determination of position in an n-ply extended manifoldness is reduced to n determinations of quantity, and therefore the determination of position in a given manifoldness is reduced to a finite number of determinations of quantity *when this is possible.* There are manifoldnesses in which the determination of position requires not a finite number, but either an endless series or a continuous manifoldness of determinations of quantity. Such manifoldnesses are, for example, the possible determinations of a function for a given region, the possible shapes of a solid figure, &c.

II. *Measure-relations of which a manifoldness of* n *dimensions is capable on the assumption that lines have a length independent of position, and consequently that every line may be measured by every other.*

Having constructed the notion of a manifoldness of n dimensions, and found that its true character consists in the property that the determination of position in it may be reduced to n determinations of magnitude, we come to the second of the problems proposed above, viz. the study of the measure-relations of which such a manifoldness is capable, and of the conditions which suffice to determine them. These measure-relations can only be studied in abstract notions of quantity, and their dependence on one another can only be represented by formulæ. On

certain assumptions, however, they are decomposable into relations which, taken separately, are capable of geometric representation; and thus it becomes possible to express geometrically the calculated results. In this way, to come to solid ground, we cannot, it is true, avoid abstract considerations in our formulæ, but at least the results of calculation may subsequently be presented in a geometric form. The foundations of these two parts of the question are established in the celebrated memoir of Gauss, *Disquisitiones generales circa superficies curvas.*

§ 1. Measure-determinations require that quantity should be independent of position, which may happen in various ways. The hypothesis which first presents itself, and which I shall here develop, is that according to which the length of lines is independent of their position, and consequently every line is measurable by means of every other. Position-fixing being reduced to quantity-fixings, and the position of a point in the n-dimensioned manifoldness being consequently expressed by means of n variables $x_1, x_2, x_3, \ldots x_n$, the determination of a line comes to the giving of these quantities as functions of one variable. The problem consists then in establishing a mathematical expression for the length of a line, and to this end we must consider the quantities x as expressible in terms of certain units. I shall treat this problem only under certain restrictions, and I shall confine myself in the first place to lines in which the ratios of the increments dx of the respective variables vary continuously. We may then conceive these lines broken up into elements, within which the ratios of the quantities dx may be regarded as constant; and the problem is then reduced to establishing for each point a general expression for the linear element ds starting from that point, an expression which will thus contain the quantities x and the quantities dx. I shall suppose, secondly, that the length of the linear element, to the first order, is unaltered when all the points of this element undergo the same infinitesimal displacement, which implies at the same time that if all the quantities dx are increased in the same ratio, the linear element will vary also in the same ratio.

On these suppositions, the linear element may be any homogeneous function of the first degree of the quantities dx, which is unchanged when we change the signs of all the dx, and in which the arbitrary constants are continuous functions of the quantities x. To find the simplest cases, I shall seek first an expression for manifoldnesses of $n-1$ dimensions which are everywhere equidistant from the origin of the linear element; that is, I shall seek a continuous function of position whose values distinguish them from one another. In going outwards from the origin, this must either increase in all directions or decrease in all directions; I assume that it increases in all directions, and therefore has a minimum at that point. If, then, the first and second differential coefficients of this function are finite, its first differential must vanish, and the second differential cannot become negative; I assume that it is always positive. This differential expression, then, of the second order remains constant when ds remains constant, and increases in the duplicate ratio when the dx, and therefore also ds, increase in the same ratio; it must therefore be ds^2 multiplied by a constant, and consequently ds is the square root of an always positive integral homogeneous function of the second order of the quantities dx, in which the coefficients are continuous functions of the quantities x. For Space, when the position of points is expressed by rectilinear co-ordinates, $ds = \sqrt{\Sigma (dx)^2}$; Space is therefore included in this simplest case. The next case in simplicity includes those manifoldnesses in which the line-element may be expressed as the fourth root of a quartic differential expression. The investigation of this more general kind would require no really different principles, but would take considerable time and throw little new light on the theory of space, especially as the results cannot be geometrically expressed; I restrict myself, therefore, to those manifoldnesses in which the line-element is expressed as the square root of a quadric differential expression. Such an expression we can transform into another similar one if we substitute for the n independent variables functions of n new independent variables. In this way, however, we cannot transform any expression into any other; since the expression contains $\frac{1}{2} n (n+1)$ coefficients which are

arbitrary functions of the independent variables; now by the introduction of new variables we can only satisfy n conditions, and therefore make no more than n of the coefficients equal to given quantities. The remaining $\frac{1}{2}n(n-1)$ are then entirely determined by the nature of the continuum to be represented, and consequently $\frac{1}{2}n(n-1)$ functions of positions are required for the determination of its measure-relations. Manifoldnesses in which, as in the Plane and in Space, the line-element may be reduced to the form $\sqrt{\Sigma dx^2}$, are therefore only a particular case of the manifoldnesses to be here investigated; they require a special name, and therefore these manifoldnesses in which the square of the line-element may be expressed as the sum of the squares of complete differentials I will call *flat*. In order now to review the true varieties of all the continua which may be represented in the assumed form, it is necessary to get rid of difficulties arising from the mode of representation, which is accomplished by choosing the variables in accordance with a certain principle.

§ 2. For this purpose let us imagine that from any given point the system of shortest lines going out from it is constructed; the position of an arbitrary point may then be determined by the initial direction of the geodesic in which it lies, and by its distance measured along that line from the origin. It can therefore be expressed in terms of the ratios dx_0 of the quantities dx in this geodesic, and of the length s of this line. Let us introduce now instead of the dx_0 linear functions dx of them, such that the initial value of the square of the line-element shall equal the sum of the squares of these expressions, so that the independent variables are now the length s and the ratios of the quantities dx. Lastly, take instead of the dx quantities $x_1, x_2, x_3, \ldots x_n$ proportional to them, but such that the sum of their squares $= s^2$. When we introduce these quantities, the square of the line-element is Σdx^2 for infinitesimal values of the x, but the term of next order in it is equal to a homogeneous function of the second order of the $\frac{1}{2}n(n-1)$ quantities $(x_1dx_2 - x_2dx_1)$, $(x_1dx_3 - x_3dx_1)\ldots$ an infinitesimal, therefore, of the fourth order; so that we obtain a finite quantity on dividing

this by the square of the infinitesimal triangle, whose vertices are $(0, 0, 0, \ldots)$, $(x_1, x_2, x_3, \ldots)$, $(dx_1, dx_2, dx_3, \ldots)$. This quantity retains the same value so long as the x and the dx are included in the same binary linear form, or so long as the two geodesics from 0 to x and from 0 to dx remain in the same surface-element; it depends therefore only on place and direction. It is obviously zero when the manifold represented is flat, *i.e.*, when the squared line-element is reducible to Σdx^2, and may therefore be regarded as the measure of the deviation of the manifoldness from flatness at the given point in the given surface-direction. Multiplied by $-\frac{3}{4}$ it becomes equal to the quantity which Privy Councillor Gauss has called the total curvature of a surface. For the determination of the measure-relations of a manifoldness capable of representation in the assumed form we found that $\frac{1}{2}n(n-1)$ place-functions were necessary; if, therefore, the curvature at each point in $\frac{1}{2}n(n-1)$ surface-directions is given, the measure-relations of the continuum may be determined from them—provided there be no identical relations among these values, which in fact, to speak generally, is not the case. In this way the measure-relations of a manifoldness in which the line-element is the square root of a quadric differential may be expressed in a manner wholly independent of the choice of independent variables. A method entirely similar may for this purpose be applied also to the manifoldness in which the line-element has a less simple expression, *e.g.*, the fourth root of a quartic differential. In this case the line-element, generally speaking, is no longer reducible to the form of the square root of a sum of squares, and therefore the deviation from flatness in the squared line-element is an infinitesimal of the second order, while in those manifoldnesses it was of the fourth order. This property of the last-named continua may thus be called flatness of the smallest parts. The most important property of these continua for our present purpose, for whose sake alone they are here investigated, is that the relations of the twofold ones may be geometrically represented by surfaces, and of the morefold ones may be reduced to those of the surfaces included in them; which now requires a short further discussion.

§ 3. In the idea of surfaces, together with the intrinsic measure-relations in which only the length of lines on the surfaces is considered, there is always mixed up the position of points lying out of the surface. We may, however, abstract from external relations if we consider such deformations as leave unaltered the length of lines—*i.e.*, if we regard the surface as bent in any way without stretching, and treat all surfaces so related to each other as equivalent. Thus, for example, any cylindrical or conical surface counts as equivalent to a plane, since it may be made out of one by mere bending, in which the intrinsic measure-relations remain, and all theorems about a plane—therefore the whole of planimetry—retain their validity. On the other hand they count as essentially different from the sphere, which cannot be changed into a plane without stretching. According to our previous investigation the intrinsic measure-relations of a twofold extent in which the line-element may be expressed as the square root of a quadric differential, which is the case with surfaces, are characterised by the total curvature. Now this quantity in the case of surfaces is capable of a visible interpretation, viz., it is the product of the two curvatures of the surface, or multiplied by the area of a small geodesic triangle, it is equal to the spherical excess of the same. The first definition assumes the proposition that the product of the two radii of curvature is unaltered by mere bending; the second, that in the same place the area of a small triangle is proportional to its spherical excess. To give an intelligible meaning to the curvature of an n-fold extent at a given point and in a given surface-direction through it, we must start from the fact that a geodesic proceeding from a point is entirely determined when its initial direction is given. According to this we obtain a determinate surface if we prolong all the geodesics proceeding from the given point and lying initially in the given surface-direction; this surface has at the given point a definite curvature, which is also the curvature of the n-fold continuum at the given point in the given surface-direction.

§ 4. Before we make the application to space, some considerations about flat manifoldnesses in general are necessary;

i.e., about those in which the square of the line-element is expressible as a sum of squares of complete differentials.

In a flat n-fold extent the total curvature is zero at all points in every direction; it is sufficient, however (according to the preceding investigation), for the determination of measure-relations, to know that at each point the curvature is zero in $\frac{1}{2}n(n-1)$ independent surface-directions. Manifoldnesses whose curvature is constantly zero may be treated as a special case of those whose curvature is constant. The common character of these continua whose curvature is constant may be also expressed thus, that figures may be moved in them without stretching. For clearly figures could not be arbitrarily shifted and turned round in them if the curvature at each point were not the same in all directions. On the other hand, however, the measure-relations of the manifoldness are entirely determined by the curvature; they are therefore exactly the same in all directions at one point as at another, and consequently the same constructions can be made from it: whence it follows that in aggregates with constant curvature figures may have any arbitrary position given them. The measure-relations of these manifoldnesses depend only on the value of the curvature, and in relation to the analytic expression it may be remarked that if this value is denoted by α, the expression for the line-element may be written

$$\frac{1}{1+\frac{1}{4}\alpha\Sigma x^2}\sqrt{\Sigma dx^2}.$$

§ 5. The theory of *surfaces* of constant curvature will serve for a geometric illustration. It is easy to see that surfaces whose curvature is positive may always be rolled on a sphere whose radius is unity divided by the square root of the curvature; but to review the entire manifoldness of these surfaces, let one of them have the form of a sphere and the rest the form of surfaces of revolution touching it at the equator. The surfaces with greater curvature than this sphere will then touch the sphere internally, and take a form like the outer portion (from the axis) of the surface of a ring; they may be rolled upon zones of spheres having less radii, but will go round

more than once. The surfaces with less positive curvature are obtained from spheres of larger radii, by cutting out the lune bounded by two great half-circles and bringing the section-lines together. The surface with curvature zero will be a cylinder standing on the equator; the surfaces with negative curvature will touch the cylinder externally and be formed like the inner portion (towards the axis) of the surface of a ring. If we regard these surfaces as *locus in quo* for surface-regions moving in them, as Space is *locus in quo* for bodies, the surface-regions can be moved in all these surfaces without stretching. The surfaces with positive curvature can always be so formed that surface-regions may also be moved arbitrarily about upon them without *bending*, namely (they may be formed) into sphere-surfaces; but not those with negative curvature. Besides this independence of surface-regions from position there is in surfaces of zero curvature also an independence of *direction* from position, which in the former surfaces does not exist.

III. *Application to Space.*

§ 1. By means of these inquiries into the determination of the measure-relations of an n-fold extent the conditions may be declared which are necessary and sufficient to determine the metric properties of space, if we assume the independence of line-length from position and expressibility of the line-element as the square root of a quadric differential, that is to say, flatness in the smallest parts.

First, they may be expressed thus: that the curvature at each point is zero in three surface-directions; and thence the metric properties of space are determined if the sum of the angles of a triangle is always equal to two right angles.

Secondly, if we assume with Euclid not merely an existence of lines independent of position, but of bodies also, it follows that the curvature is everywhere constant; and then the sum of the angles is determined in all triangles when it is known in one.

Thirdly, one might, instead of taking the length of lines to be independent of position and direction, assume also an independence of their length and direction from position. According to this conception changes or differences of position are complex magnitudes expressible in three independent units.

§ 2. In the course of our previous inquiries, we first distinguished between the relations of extension or partition and the relations of measure, and found that with the same extensive properties, different measure-relations were conceivable; we then investigated the system of simple size-fixings by which the measure-relations of space are completely determined, and of which all propositions about them are a necessary consequence; it remains to discuss the question how, in what degree, and to what extent these assumptions are borne out by experience. In this respect there is a real distinction between mere extensive relations, and measure-relations; in so far as in the former, where the possible cases form a discrete manifoldness, the declarations of experience are indeed not quite certain, but still not inaccurate; while in the latter, where the possible cases form a continuous manifoldness, every determination from experience remains always inaccurate: be the probability ever so great that it is nearly exact. This consideration becomes important in the extensions of these empirical determinations beyond the limits of observation to the infinitely great and infinitely small; since the latter may clearly become more inaccurate beyond the limits of observation, but not the former.

In the extension of space-construction to the infinitely great, we must distinguish between *unboundedness* and *infinite extent*, the former belongs to the extent relations, the latter to the measure-relations. That space is an unbounded three-fold manifoldness, is an assumption which is developed by every conception of the outer world; according to which every instant the region of real perception is completed and the possible positions of a sought object are constructed, and which by these applications is for ever confirming itself. The unboundedness of space possesses in this way a greater empirical certainty than any external experience. But its infinite extent by no means

follows from this; on the other hand if we assume independence of bodies from position, and therefore ascribe to space constant curvature, it must necessarily be finite provided this curvature has ever so small a positive value. If we prolong all the geodesics starting in a given surface-element, we should obtain an unbounded surface of constant curvature, *i.e.*, a surface which in a *flat* manifoldness of three dimensions would take the form of a sphere, and consequently be finite.

§ 3. The questions about the infinitely great are for the interpretation of nature useless questions. But this is not the case with the questions about the infinitely small. It is upon the exactness with which we follow phenomena into the infinitely small that our knowledge of their causal relations essentially depends. The progress of recent centuries in the knowledge of mechanics depends almost entirely on the exactness of the construction which has become possible through the invention of the infinitesimal calculus, and through the simple principles discovered by Archimedes, Galileo, and Newton, and used by modern physic. But in the natural sciences which are still in want of simple principles for such constructions, we seek to discover the causal relations by following the phenomena into great minuteness, so far as the microscope permits. Questions about the measure-relations of space in the infinitely small are not therefore superfluous questions.

If we suppose that bodies exist independently of position, the curvature is everywhere constant, and it then results from astronomical measurements that it cannot be different from zero; or at any rate its reciprocal must be an area in comparison with which the range of our telescopes may be neglected. But if this independence of bodies from position does not exist, we cannot draw conclusions from metric relations of the great, to those of the infinitely small; in that case the curvature at each point may have an arbitrary value in three directions, provided that the total curvature of every measurable portion of space does not differ sensibly from zero. Still more complicated relations may exist if we no longer suppose the linear element expressible as the square root of a quadric differential.

Now it seems that the empirical notions on which the metrical determinations of space are founded, the notion of a solid body and of a ray of light, cease to be valid for the infinitely small. We are therefore quite at liberty to suppose that the metric relations of space in the infinitely small do not conform to the hypotheses of geometry; and we ought in fact to suppose it, if we can thereby obtain a simpler explanation of phenomena.

The question of the validity of the hypotheses of geometry in the infinitely small is bound up with the question of the ground of the metric relations of space. In this last question, which we may still regard as belonging to the doctrine of space, is found the application of the remark made above; that in a discrete manifoldness, the ground of its metric relations is given in the notion of it, while in a continuous manifoldness, this ground must come from outside. Either therefore the reality which underlies space must form a discrete manifoldness, or we must seek the ground of its metric relations outside it, in binding forces which act upon it.

The answer to these questions can only be got by starting from the conception of phenomena which has hitherto been justified by experience, and which Newton assumed as a foundation, and by making in this conception the successive changes required by facts which it cannot explain. Researches starting from general notions, like the investigation we have just made, can only be useful in preventing this work from being hampered by too narrow views, and progress in knowledge of the interdependence of things from being checked by traditional prejudices.

This leads us into the domain of another science, of physic, into which the object of this work does not allow us to go to-day.

Synopsis.

PLAN of the Inquiry:

I. Notion of an n-ply extended magnitude.

§ 1. Continuous and discrete manifoldnesses. Defined parts of a manifoldness are called Quanta. Division of the theory of continuous magnitude into the theories,

(1) Of mere region-relations, in which an independence of magnitudes from position is not assumed;

(2) Of size-relations, in which such an independence must be assumed.

§ 2. Construction of the notion of a one-fold, two-fold, n-fold extended magnitude.

§ 3. Reduction of place-fixing in a given manifoldness to quantity-fixings. True character of an n-fold extended magnitude.

II. Measure-relations of which a manifoldness of n-dimensions is capable on the assumption that lines have a length independent of position, and consequently that every line may be measured by every other.

§ 1. Expression for the line-element. Manifoldnesses to be called Flat in which the line-element is expressible as the square root of a sum of squares of complete differentials.

§ 2. Investigation of the manifoldness of n-dimensions in which the line-element may be represented as the square root of a quadric differential. Measure of its deviation from flatness (curvature) at a given point in a given surface-direction. For the determination of its measure-relations it is allowable and sufficient that the curvature be arbitrarily given at every point in $\frac{1}{2}n(n-1)$ surface directions.

§ 3. Geometric illustration.

§ 4. Flat manifoldnesses (in which the curvature is everywhere = 0) may be treated as a special case of manifoldnesses with constant curvature. These can also be defined as admitting an independence of n-fold extents in them from position (possibility of motion without stretching).

§ 5. Surfaces with constant curvature.

III. Application to Space.

§ 1. System of facts which suffice to determine the measure-relations of space assumed in geometry.

§ 2. How far is the validity of these empirical determinations probable beyond the limits of observation towards the infinitely great?

§ 3. How far towards the infinitely small? Connection of this question with the interpretation of nature.

X.

ANALOGUES OF PASCAL'S THEOREM*.

1. A SYSTEM of $2n$ right lines will in general have $n(2n-1)$ intersections. If we divide them into two systems containing n lines each, and only consider the intersections of one system by the other, the number is reduced to n^2. Suppose further that $n(n-1)$ of these lie on a curve of the $(n-1)$th degree. Then we call the system a $2n$-lateral $n(n-1)$-gon totally inscribed in the $(n-1)^{\text{ic}}$. The $n(n-1)$ points which lie on the curve we call the *ineunts* of the $2n$-lateral; and the other n intersections we call the *exeunts*. The $2n$-lateral is said to be *totally* inscribed, because all the points in which any side is cut by the curve are ineunts of the figure. Such a figure we denote by the square symbol

$$\left\{\begin{array}{llll} 1, & A_1, & B_1, & C_1 \ldots\ldots N_1 \\ N_2, & 2, & A_2, & B_2 \ldots\ldots \\ M_3, & N_3, & 3, & A_3 \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \end{array}\right\} \ldots\ldots\ldots\ldots (1),$$

where the rows denote one system of n right lines, and the columns the other; and the n exeunts are ranged on the diagonal line from the left-hand upper corner to the right-hand lower corner. A_1, B_1, ... &c. are the ineunts.

We propose, first, to prove in various ways that the n exeunts always lie on a right line; and, secondly, to demon-

* [From *The Quarterly Journal of Pure and Applied Mathematics*, No. 23, March, 1864, pp. 216—222.]

strate one or two other properties of the figure. The present communication is confined for the most part to the case in which $n=4$; in a second we hope to notice some peculiarities of the higher plane cases, and to state the *true* analogues of Pascal's Theorem in Geometry of Higher Dimensions

2. *Of the mn-intersections of two curves of the* mth *and* nth *degrees, if* pn *lie on a curve of the* pth *degree, the remaining* $(m-p)\,n$ *will lie on a curve of the* $(m-p)$th *degree* (p being less than m, and n not greater than m).

Let U_m, U_n, U_p be three curves of degrees m, n, p, respectively. Since U_m passes through all the intersections of U_n and U_p, U_m must be of the form

$$\phi U_n + \psi U_p \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2);$$

but U_m is of the mth degree and U_p of the pth, therefore ψ must be of the degree $m-p$; but $\psi=0$ passes through all the remaining intersections of U_m and U_n. Therefore, &c. Q.E.D.

This includes (when $m=n$) the well known theorem of Salmon's *Higher Plane Curves*, Chap. II. Sect. 1, Art. 24. And the latter at once gives the property that the n exeunts of (1) lie on a right line. For we have two systems of n right lines, and a curve of the $(n-1)$th degree passing through $n(n-1)$ of their intersections; therefore the other n intersections lie on a right line. In its general form the theorem shews that we may erase any of the rows of the symbol (1) (which we call the "*index*" of the figure) without destroying the property of the exeunts.

3. We shall have occasion to use Prof. Cayley's theorem, that "*every curve of the* mth *order,* (m not being less than n or p, nor greater than $n+p-3$), *which passes through all but* $\frac{1}{2}(n+p-m-1)(n+p-m-2)$ *of the intersections of two curves of the* nth *and* pth *orders, passes also through the remaining intersections.*" The form (2) contains $\frac{1}{2}(m-n)(m-n+3)$ constants in ϕ, $\frac{1}{2}(m-p)(m-p+3)$ in ψ, and one constant of mul-

tiplication. Therefore it is the general equation of a curve of the mth degree through

$$\Phi \equiv \tfrac{1}{2} m(m+3) - \tfrac{1}{2}(m-n)(m-n+3) - \tfrac{1}{2}(m-p)(m-p+3) - 1$$

points of intersection of two curves of the nth and pth degrees. It is easily shewn that

$$np - \Phi \equiv \tfrac{1}{2}(n+p-m-1)(n+p-m-2),$$

and the equation (2) evidently represents a curve passing through *all* the intersections of U_n and U_p. The limitations to the value of m are obvious.

4. We now consider the case of an octolateral dodecagon inscribed in a cubic. Given two sets of four right lines a, b, c, d, and $\alpha, \beta, \gamma, \delta$, it follows from the last proposition that a cubic through eleven of their intersections will pass through a twelfth ($m = n = 4$, $p = 3$). Take then any straight line a meeting the cubic in three points A, B, C; through these draw straight lines β, γ, δ, meeting the curve in A, K, M; B, D, N; C, E, G respectively; join DE, GK, MN by straight lines b, c, d, meeting the curve again in F, H, L respectively; then F, H, L lie on a right line α. For, in the first place, join F, H by a line α; then the quartic $(\alpha\beta\gamma\delta)$ passes through all but one of the intersections of the quartic $(abcd)$ with the given cubic, and so it must pass through the remaining one, or L lies on FH. We write down the "index" of this figure

$$\left\{\begin{array}{cccc} 1, & A, & B, & C \\ F, & 2, & D, & E \\ H, & K, & 3, & G \\ L, & M, & N, & 4 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Here the rows are the straight lines a, b, c, d, and the columns are $\alpha, \beta, \gamma, \delta$. The points $a\alpha$, $b\beta$, $c\gamma$, $d\delta$ are the exeunts 1 2 3 4, which lie on a right line.

5. We give another proof on account of its consequences. With the notation of *Higher Plane Curves*, Chap. III., Sec. 6,

we form the equation $A + B + C = 0$ to express that the points A, B, C lie on a right line, and so for the rest. Now let us take a as an abbreviation for the *quantity* $A + B + C$, and so on. Thus $a + b + c + d$ includes all the twelve points, and so does $\alpha + \beta + \gamma + \delta$. Hence

$$a + b + c + d \equiv \alpha + \beta + \gamma + \delta,$$

and therefore if seven of these quantities vanish, the eighth must vanish also. But we know that $a, b, c, d, \beta, \gamma, \delta$ all $= 0$, therefore $\alpha = 0$ also, or FHL is a straight line.

6. *An octolateral has six diagonals which intersect by pairs in three points of the curve lying on a right line.*

With the notation of the last article, each of the quantities a, b, c, d; $\alpha, \beta, \gamma, \delta$ is identically zero. So therefore is any quantity formed from them by addition or subtraction. Now consider the identities

$$\left.\begin{aligned}(a+b+\alpha+\beta)-(c+d+\gamma+\delta) &\equiv (A+F)-(N+G)\\ (a+c+\alpha+\gamma)-(b+d+\beta+\delta) &\equiv (B+H)-(E+M)\\ (a+d+\alpha+\delta)-(b+c+\beta+\gamma) &\equiv (C+L)-(D+K)\end{aligned}\right\}\ldots\ldots(4).$$

If AF meet the curve in P, so that $A + F + P = 0$, we must have also $N + G + P = 0$, since $A + F = N + G$ identically. That is, AF and NG meet the curve in the same point. Similarly for BH and EM, CL and DK. Call these points P, Q, R. Then, if we form the six equations like $A + F + P = 0$, and add them all together, we have

$$2(P + Q + R) + A + B + C + D + \ldots = 0;$$

therefore $P + Q + R = 0$, or PQR is a straight line.

7. By aid of this property we may consider the octolateral from an entirely different point of view.

The cubic passes through all the intersections of AF, γ, δ with NG, a, b. Therefore its equation may be written

$$AF \cdot \gamma\delta + NG \cdot ab = 0 \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

where, for shortness, AF represents the perpendicular from a current point on AF, so that $AF=0$ is the equation to AF. A constant multiplier is of course supposed.

Now if we consider this equation generally, it contains *thirteen* independent constants, viz. two for each of the six lines, and one constant of multiplication. But the general equation of the third degree contains *nine* constants; we may therefore, as the cubic is given, assume *four* points in (5) and the rest will be determined. Assume then P, the intersection of AF and NG, and the points A, N. These determine the straight lines PAF, PNG, and we have still one point at our disposal. Choose any point B for the intersection of a and γ; join AB, which is a, and then C, the third point where it cuts the curve, will be a point on δ. To find γ, δ we have a choice of two constructions; we may join either BN, CG, or BG, CN; we adopt the former. If BN meet the curve in D, and CG in E, then the form of equation (5) shews that FDE is a straight line, namely, b. We have constructed then the following portion of the index (3),

$$\left\{\begin{matrix} *, & A, & B, & C \\ F, & *, & D, & E \\ & & *, & G \\ & & N, & * \end{matrix}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

The point P is of course not represented.

Now choose another point M, and use it as B was used before to obtain three more points H, K, L corresponding to ECD. We have then a second equation to the cubic

$$AF \,.\, cd + NG \,.\, \alpha\beta = 0 \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

and we may complete the symbol (6) in the form

$$\left\{\begin{matrix} 1, & A, & B, & C \\ F, & 2, & D, & E \\ H, & K, & 3, & G \\ L, & M, & N, & 4 \end{matrix}\right\},$$

the exeunts 1 2 3 4 being now determined.

8. *The four exeunts lie on a right line.*

We write the equivalent equations (5) and (7) in the forms

$$-\frac{AF}{NG}=\frac{ab}{\gamma\delta}=\frac{\alpha\beta}{cd} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8),$$

and as these are satisfied for every point on the cubic, it follows that the equation

$$abcd-\alpha\beta\gamma\delta=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

must have the cubic as a factor. But (9) is of the fourth degree; therefore the other factor must be of the first degree, which represents a straight line.

Or we may arrange the proof as follows: we have

$$AF.\gamma\delta+NG.ab\equiv lU=0,$$

and

$$AF.cd+NG.\alpha\beta\equiv mU=0;$$

therefore

$$AF(p.\gamma\delta+q.cd)+NG\,(p.ab+q.\alpha\beta)\equiv(pl+qm)\,U=0.$$

Now suppose $p:q$ so taken that $(p.\gamma\delta+q.cd)$ may represent the pair of lines NG, 3 4. Then NG is a factor of the left-hand side of the identity, but not of the right; therefore both sides must vanish identically. Thus we have

$$AF.NG.34\equiv-NG\,(p.ab+q.\alpha\beta),$$

but, the left-hand side of this identity denoting three right lines, the right must also. But these can only be NG, AF, 1 2; therefore 1 2 is identical with 3 4, or 1 2 3 4 is a straight line. This straight line we call the *axis* of the octolateral.

9. The method of (7) presents an octolateral as the aggregate of a pair of diagonals Δ, and two quadrangles, X, Y, formed from Δ. Suppose we form from Δ another quadrangle Z; then we have three octolaterals, ΔXY, ΔYZ, ΔZX, and *the*

axes of these three meet in a point. For we write down the equations

$$AF.\gamma\delta + NG.ab = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (X),$$
$$AF.cd + NG.\alpha\beta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (Y),$$
$$AF.xy + NG.zw = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (Z),$$
$$-\frac{NG}{AF} = \frac{xy}{zw} = \frac{cd}{\alpha\beta} = \frac{\gamma\delta}{ab}.$$

The last statement expressing the property in question.

10. The proposition proved in (6) may be extended. *Any six diagonals which include all the ineunts meet the cubic in six new points which lie on a conic.* This is proved precisely as in (6), the addition of six equations giving the result

$$P + Q + R + S + T + U = 0,$$

which is the condition. The number of these conics we have not had the courage to count. There are nine quadrangles of ineunts which yield three pairs of diagonals each, and by combinations of these we obtain 96 conics, including the straight line PQR, Art. 6. But there are a great many other arrangements which cannot be so divided into quadrangles. The diagonals may be classified; there are 6 of the class $\alpha\beta.b\alpha$, 12 of the class $a\beta.c\delta$, and 24 of the class $a\beta.b\gamma$. The property of the six points of intersection of a conic and a cubic is this: *if they be divided into three pairs, the lines joining these will meet the cubic in three points on a right line.* For since

$$P + Q + R + S + T + U = 0,$$

if

$$P + Q + X = 0,$$
$$R + S + Y = 0,$$
$$T + U + Z = 0,$$

we must have $X + Y + Z = 0$. *To every such six points pertain* 15 *of these lines;* for if we take any pair PQ, the remaining four may be divided into pairs in three ways, RS, TU; RT, SU; RU, ST; and so for each of the pairs PR, PS, PT, PU. Suppose now that P, Q are the points where the diagonals

BE, KL meet the curve again; and let BK, EL meet it in P_1, Q_1, and BL, EK in P_2, Q_2. Then $R+S+T+U$ is constant, while the sum of the six is always zero; thus

$$P+Q=P_1+Q_1=P_2+Q_2,$$

or *the lines* PQ, P_1Q_1, P_2Q_2, *meet on the curve.* If then we were counting all the lines XYZ, we must [not*] take the number of conics and multiply it by 15.

11. If at any point A we draw a tangent AB, then $2A+B=0$. Prof. Cayley calls the point B the *Satellite of* A. It is easily seen that the Satellites of an octolateral form another octolateral, and that any points connected with the second are Satellites of points similarly connected with the first. Many other properties may easily be proved.

Nov. 20, 1863.

* [*Not* is introduced on the authority of a pencilled correction in Prof. Clifford's own handwriting.]

XI.

ANALYTICAL METRICS*.

I. *Introduction.*

1. ANY one must have observed that there are two kinds of theorems in Geometry; one kind having reference to *position* only, the other kind having reference to *magnitude.* Pascal's theorem is an example of the first, or *graphic* geometry; Euclid I. 47 is an example of the second, or *metric* geometry. It may be possible to state the same theorem in two ways, so as to make it either metric or graphic. In such a case the graphic statement may be distinguished by the fact that it is *unaltered by projection.* In fact, the word *graphic* is co-extensive with *projective.* And so, bearing in mind the properties of projection, we may define Metric Geometry as that science which has to do with the magnitudes of angles, distances, areas, and volumes. It seems, at first sight, as if the method of coordinates, especially in its more complete homogeneous form, were almost wholly applicable to Graphic Geometry, and altogether unfit for the study of Metrics. And this idea is strengthened by the fact that nearly the whole of Graphic Geometry is due to the method of coordinates, while the science of Metrics has hitherto benefitted by it very little indeed. But there are two considerations which go against this view. The first is derived from Poncelet's discovery of the true nature of metrics. Poncelet shewed that all circles pass through the same two points at infinity, and that

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all angles, lengths, &c. may be expressed as *graphic* functions of these two points. For example, the angle between two straight lines is a certain function of the anharmonic ratio in which they cut the line joining the circular points. The general principle may be thus stated:—Whenever we speak of the metric properties of loci, we consider the loci, not by themselves, but in connection with another *fixed* locus, called the Absolute. Thus, in the case of Plane Geometry, the Absolute is the two circular points at infinity.

The second consideration is derived from the Higher Algebra. M. Magnus has proved that linear transformation is virtually equivalent to projection, and that graphic properties are in fact those which are unaltered by linear transformation. Now it is one of the propositions of the Higher Algebra, that when we consider any set of loci, the invariants, or functions of the coefficients, and covariants, or connected loci, which are unaltered by linear transformation, are *limited in number*. All graphic properties, therefore, may be stated in terms of a *finite number* of expressions. Now combine these considerations; first, that Metric Geometry may be analytically reduced to Graphic; and, secondly, that Graphic Geometry is necessarily of finite extent and exhaustible; and it will, I think, be abundantly evident that Analytical Metrics ought to be studied.

As to the mode of proof, I have freely made use of known results both in Pure Geometry and in Cartesian Coordinates, though the formulæ are to be regarded as referring primarily to systems of coordinates in which the equations are homogeneous. It is, of course, possible to *start* Coordinate Geometry from definitions or axioms, without referring at all to any other science, such as Pure Geometry; and in this way one may arrive at the results even of Metrics. But the course here adopted seemed to be, in existing circumstances, shorter. Those who wish to see the subject handled scientifically, must refer to Prof. Cayley's Sixth Memoir upon Quantics.

2. I give here two sample modes of proof, which together will suffice to prove most of the propositions which follow. Proofs which are similar to these will hereafter be omitted.

(*A*) *To find the area of the triangle contained by three points whose equations are given in the form*

$$lx + my + nz = 0,$$

(the area of the fundamental triangle being 1).

The area vanishes only when the points are in a straight line, and becomes infinite only when one of them is at infinity. Now the condition that they shall be in a straight line is

$$J \equiv \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} = 0,$$

and the product of the conditions that they shall be at infinity is

$$P \equiv (l_1 + m_1 + n_1)\,(l_2 + m_2 + n_2)\,(l_3 + m_3 + n_3).$$

Now the expression for the area must be of no dimensions in the coefficients of either point; and when two points and the area are given, the locus of the other is a straight line. The required expression is therefore $m \cdot \frac{J}{P}$ where m is a constant. By reference to the fundamental triangle, we get $m = 1$; therefore the area is $\frac{J}{P}$.

(*B*) *To find the cosine of the angle between two straight lines* $A = 0$, $B = 0$.

Let $\phi(A) = 0$ be the condition that A pass through one of the circular points; we know from Cartesians that this is of the second order in the coefficients of A. And because it is of the second order $\phi(\lambda A + \mu B)$ must be of the form

$$\lambda^2 \phi A + 2\lambda\mu\psi AB + \mu^2 \phi B \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now when we put for A and B their Cartesian equivalents, the coefficients of powers of λ, μ in (1) cannot alter in ratio, because the evanescence of (1) gives the two values of $\lambda : \mu$, for which $\lambda A + \mu B = 0$ represents a line through one of the circular points, and these lines are the same in both cases. But in Cartesian coordinates, the expression $\frac{\psi AB}{\sqrt{(\phi A \,.\, \phi B)}}$ is equal to the cosine of

the angle between A and B; this is therefore its value in all other systems of coordinates.

The function ϕ is called the Absolute, and the determination of its form is an important part of our subject.

NOTE. (α) The function used in proof (A), and called J, will occur very often in the sequel. Suppose we have a number of linear equations equal to the number of variables in each, and that we form the determinant which is the result of eliminating the variables from these equations; then this determinant is called their Jacobian, and will be denoted by $J(LMN\ldots)$, where $L=0$, $M=0$, &c., ... are the linear equations.

(β) The abbreviation $\infty=0$ will be used to represent the equation to the line, &c. at infinity; so that $J(LM\infty)$ means the result of eliminating the variables between $L=0$, $M=0$, and the equation to the line at infinity.

(γ) In the demonstration (B) it is important to remember, that when we put for A and B their Cartesian equivalents, these expressions are transformed by the *same* substitution; so that if A is changed to A', and B to B', $lA+mB$ will be changed to $lA'+mB'$. The proof depends on this. It would not be sufficient to know that $A=0$, in a homogeneous system, and $A'=0$, in Cartesians, represent the same straight line, because then a constant factor might be introduced, of which we should know nothing.

(δ) The method of comparison of dimensions, made use of almost exclusively in the following sections, requires a little explanation. In the first place, all the functions employed are integral functions; so that it is quite legitimate to say "Because A always vanishes when B does, therefore A contains B as a factor." Secondly, every expression must contain some factor depending on the coefficients of the "Absolute;" but these factors will be systematically disregarded, as they are only numerical, and can easily be determined by reference to any simple particular case. These things being so, the method of comparison of dimensions amounts simply to this; when we have proved the existence of factors enough to account for all the dimensions

of an expression, we may be sure that it has no other factor. And, an analytical equivalence being thus proved, it only remains to divide both sides by such quantities as will make it interpretable geometrically.

II. *Geometry of One Dimension. Graphometrics.*

3. Geometry of one dimension treats of a series of points on a straight line, which are determined by the ratios of their distances from two fixed points $xy = 0$, given by equations of the form $lx + my = 0$. The equation to the point at infinity on the line is clearly $x - y = 0$; and so the condition that the point $lx + my$ may be at infinity is $l + m = 0$. The condition that two points $A \equiv l_1x + m_1y$ and $B \equiv l_2x + m_2y$ may be coincident is

$$J(AB) \equiv \begin{vmatrix} l_1, & l_2 \\ m_1, & m_2 \end{vmatrix} = 0.$$

So the condition that A may be at infinity may be written $J(A\infty) = 0$.

The distance between A and B is $\dfrac{J(AB)}{J(A\infty).J(B\infty)}$. (Proof *A*.)

Graphometrics.

4. Let A, B, C, D be four points in a straight line; then it appears, by Art. 3, that

$$\frac{AB.CD}{AC.DB} = \frac{J(AB).J(CD)}{J(AC).J(DB)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Now it is known that in linear transformation the function $J(AB)$ is only affected with a constant numerical factor. This factor disappears in the expression (2); the expression, therefore, is unaltered by linear transformation. It is, in fact, an anharmonic ratio of the points $ABCD$.

Now this function belongs to Metric Geometry, because it is expressed by means of *lengths*, viz. $\dfrac{AB.CD}{AC.DB}$. But it also belongs to Graphic Geometry, because it is unaltered by projection

or linear transformation. On these accounts I propose to call it a *Graphometric* function. The reason why a new name is needed, is that there are other Graphometric functions, which are naturally presented by analytical metrics. For instance, it will be proved that if $ABCDEF$ are six points in a plane, the ratio $\frac{(ABC)\,(DEF)}{(ABD)\,(CEF)}$, between the products of the triangles they contain, is graphometric, or independent of projection though expressed in terms of areas. And it is natural to suppose that these and similar functions will be important in Solid Geometry, because the analogous function, namely, anharmonic ratio, is so important in Plane Geometry. For this reason I shall, in what follows, pay particular attention to Graphometrics.

III. *Points, Lines, and Circles.*

5. In respect of one line $L=0$, the expression considered is $\phi(L)$, whose evanescence is the condition that the line shall pass through one of the circular points. If the line be, in Cartesians, $lx+my+n=0$, then

$$\phi L \equiv l^2+m^2,$$

that is, it is of the second order in the coefficients of L, and its discriminant vanishes. We have, also,

$$\phi(\lambda L+\mu M) \equiv \lambda^2(l_1^2+m_1^2)+2\lambda\mu(l_1l_2+m_1m_2)+\mu^2(l_2^2+m_2^2);$$

or
$$\lambda^2\phi L+2\lambda\mu\psi(L,M)+\mu^2\phi M \ldots\ldots\ldots\ldots\ldots(3),$$

so that in any system of coordinates the function $\psi(LM)$, thus formed, is the condition that L and M may be at right angles. In the same way, or from the nature of the case, it is seen that the condition that (3) may be a perfect square is the square of the condition that L and M may be parallel, or may meet on the line at infinity; that is

$$\phi L.\,\phi M-(\psi LM)^2 \equiv J(LM\infty)^2 \ldots\ldots\ldots\ldots(4)^*.$$

In respect of three lines the function $J(LMN)$ will be considered.

* The equation to the line at infinity will be found from ϕ.

The principal function of a point is $(A\,\infty)$, the condition that it may be at infinity. Given two points $(l_1m_1n_1)$, $(l_2m_2n_2)$, the line joining them is

$$\begin{vmatrix} m_1, & n_1 \\ m_2, & n_2 \end{vmatrix} x + \begin{vmatrix} n_1, & l_1 \\ n_2, & l_2 \end{vmatrix} y + \begin{vmatrix} l_1, & m_1 \\ l_2, & m_2 \end{vmatrix} z = 0;$$

from this it will be easy to understand ϕAB, the condition that AB may pass through a circular point, and $\psi(AB, CD)$, the condition that AB and CD may be at right angles.

Angles.

6. The cosine of the angle between L and M has been proved to be $\dfrac{\psi LM}{\sqrt{(\phi L \,.\, \phi M)}}$; by (4) the sine is $\dfrac{J(LM\,\infty)}{\sqrt{(\phi L \,.\, \phi M)}}$.

If then we consider four lines, $LMNR$, we have

$$\frac{\sin(LM) \,.\, \sin(NR)}{\sin(LN) \,.\, \sin(RM)} \equiv \frac{J(LM\,\infty) \,.\, J(NR\,\infty)}{J(LN\,\infty) \,.\, J(RM\,\infty)}.$$

From the identity*

$$J(LM\,\infty) \,.\, J(NR\,\infty) + J(LN\,\infty) \,.\, J(RM\,\infty) + J(LR\,\infty) \,.\, J(MN\,\infty) = 0,$$

we may therefore deduce

$$\sin(LM) \,.\, \sin(NR) + \sin(LN) \,.\, \sin(RM) + \sin(LR) \,.\, \sin(MN) = 0 \ldots\ldots\ldots (5).$$

If we make N perpendicular to R, this becomes the formula

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B.$$

Next, consider the determinant $\begin{vmatrix} \psi LN, & \psi LR \\ \psi MN, & \psi MR \end{vmatrix}$.

This vanishes when L is parallel to M, for then the two rows become identical, and also when N is parallel to R, for then the

* A method of proving these identities will be given afterwards. This case may also be treated in the same way as the next.

columns become identical. Hence, by a comparison of dimensions, we can see that

$$\begin{vmatrix} \psi LN, & \psi LR \\ \psi MN, & \psi MR \end{vmatrix} \equiv J(LM\infty) \,.\, J(NR\infty),$$

or, which is the same thing,

$$\cos(LN) \,.\, \cos(MR) - \cos(LR) \,.\, \cos(MN) = \sin(LM) \,.\, \sin(NR) \quad \ldots\ldots\ldots(6).$$

If we make M parallel to R, this becomes the formula

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B.$$

Since $\psi(LM)$ is of the *first order* in the coefficients of each of the lines, we must have

$$\psi(D, lA + mB + nC) \equiv l\psi AD + m\psi BD + n\psi CD,$$

so that, if l, m, n are regarded as variables, the tangential equation to the point at infinity in a direction perpendicular to D is

$$l \,.\, \psi AD + m \,.\, \psi BD + n \,.\, \psi CD = 0.$$

We have similar equations for points so connected with E and F. But these three points, being all at infinity, are in the same straight line; we must therefore have identically

$$\begin{vmatrix} \psi AD, & \psi BD, & \psi CD \\ \psi AE, & \psi BE, & \psi CE \\ \psi AF, & \psi BF, & \psi CF \end{vmatrix} \equiv 0\,;$$

that is to say

$$\begin{vmatrix} \cos(AD), & \cos(BD), & \cos(CD) \\ \cos(AE), & \cos(BE), & \cos(CE) \\ \cos(AF), & \cos(BF), & \cos(CF) \end{vmatrix} \equiv 0 \quad \ldots\ldots\ldots(7),$$

where A, B, C, D, E, F are any six lines. By making A, B, C coincide with D, E, F respectively, we obtain the relation which exists among the cosines of the angles of a triangle; but the signs are undetermined.

The theorem (7) may also be proved indirectly as follows. Form the reciprocal determinant, expressing the minors by the help of (6); it will be found to vanish identically.

Distances.

7. The distance between two points A and B is $\frac{\sqrt{\phi AB}}{A\infty . B\infty}$.

The length of the perpendicular from a point A on a line L is $\frac{AL}{A\infty . \sqrt{\phi L}}$; where AL denotes the result of substituting the coefficients of A for the variables in L, or *vice versâ*.

So the length of a line N, cut off by two lines L and M, is $\frac{\sqrt{\phi N} . J(LMN)}{J(LN\infty) . J(MN\infty)}$.

And the length of the perpendicular from C on AB is

$$\frac{J(ABC)}{C\infty . \sqrt{\phi AB}}.$$

The point where the line AB cuts any line Π is of course represented by $\frac{A}{\Pi A} = \frac{B}{\Pi B}$. So that if L and M are two lines, the evanescence of the determinant

$$\begin{vmatrix} LA, & LB \\ MA, & MB \end{vmatrix}$$

is the condition that L, M, AB may meet in a point; in fact, by comparing dimensions, we may see that it is the same as $J(L, M, AB)$ or $J(LM, A, B)$. The interpretation of this will be given afterwards. Now let Π, Π' be two lines joining the point C to the circular points at infinity. Then it is clear that the product

$$\begin{vmatrix} \Pi A, & \Pi B \\ A\infty, & L\infty \end{vmatrix} \times \begin{vmatrix} \Pi' A, & \Pi' B \\ A\infty, & B\infty \end{vmatrix}$$

must vanish when $\phi AB = 0$, and also when $C\infty = 0$; in fact, by comparison of dimensions, it is seen to be $\phi AB . C\infty^2$. Now $\Pi A . \Pi' A = \phi AC$, and

$$\Pi A . \Pi' B + \Pi' A . \Pi B = 2\psi(AC, BC);^*$$

* Suppose A and B to be $(l_1 m_1 n_1)$ and $(l_2 m_2 n_2)$, and write Bd_A for the symbol $l_2 d_{l_1} + \ldots$, then $2\psi(AC, BC) = Bd_A . \phi AC = \Pi A . \Pi' B + \Pi' A . \Pi B$.

thus we have

$$C\infty^2 . \phi AB \equiv A\infty^2 . \phi BC + 2A\infty . B\infty . \psi(AC, BC) + B\infty^2 . \phi AC \ldots\ldots (8),$$

or $$AB^2 = BC^2 + CA^2 + 2BC . CA \cos(BC, CA).$$

By means of the identity (8) I propose to find the value of the determinant

$$\begin{vmatrix} \phi AE, & \phi BE, & \phi CE, & \phi DE, & E\infty^2 \\ \phi AF, & \phi BF, & \phi CF, & \phi DF, & F\infty^2 \\ \phi AG, & \phi BG, & \phi CG, & \phi DG, & G\infty^2 \\ \phi AH, & \phi BH, & \phi CH, & \phi DH, & H\infty^2 \\ A\infty^2, & B\infty^2, & C\infty^2, & D\infty^2, & 0 \end{vmatrix}.$$

Take any point O, and substitute for the several constituents by the formula (8). Then by subtracting the last row and column, with proper multipliers, from each of the others, we may reduce the determinant to the form

$$\begin{vmatrix} \psi(OA, OE), & \psi(OB, OE), & \psi(OC, OE), & \psi(OD, OE), & E\infty \\ \psi(OA, OF), & \psi(OB, OF), & \psi(OC, OF), & \psi(OD, OF), & F\infty \\ \psi(OA, OG), & \psi(OB, OG), & \psi(OC, OG), & \psi(OD, OG), & G\infty \\ \psi(OA, OH), & \psi(OB, OH), & \psi(OC, OH), & \psi(OD, OH), & H\infty \\ A\infty, & B\infty, & C\infty, & D\infty, & 0 \end{vmatrix}$$

multiplied by $\dfrac{A\infty . B\infty . C\infty . D\infty . E\infty . G\infty . H\infty}{O\infty^6}$. But by (7) every term of this vanishes identically. Hence we obtain the relation which connects the distances of four points, 1234, from four other points, 5678 :

$$\begin{vmatrix} 15^2, & 25^2, & 35^2, & 45^2, & 1 \\ 16^2, & 26^2, & 36^2, & 46^2, & 1 \\ 17^2, & 27^2, & 37^2, & 47^2, & 1 \\ 18^2, & 28^2, & 38^2, & 48^2, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix} \equiv 0.$$

When 1234 are identical with 5678, this gives the relation between the distances of any four points in a plane.

Triangles.

8. The area of the triangle formed by three straight lines L, M, N is the same as that formed by the points MN, NL, LM; that is

$$\frac{J(MN, NL, LM)}{MN\infty \,.\, NL\infty \,.\, LM\infty} \equiv \frac{J(LMN)^2}{J(MN\infty)\,.\,J(NL\infty)\,.\,J(LM\infty)}.$$

This I call $\frac{J^2}{P}$. I write also Π for $\phi L \,.\, \phi M \,.\, \phi N$. In this notation it is easy to see that

$$a = \frac{J}{P}\sqrt{\phi L}\,.\,J(MN\infty), \text{ \&c.}; \quad abc = \frac{J^3}{P^2}\sqrt{\Pi};$$

$$\sin A \sin B \sin C = \frac{P}{\Pi}; \quad \frac{a}{\sin A} = \frac{J}{P}\sqrt{\Pi};$$

and so on.

In the case of six *points*, 123456, it is clear that, if $\Delta\,(123)$ denote the area of the triangle 123,

$$\frac{\Delta\,(123)\,\Delta\,(456)}{\Delta\,(124)\,\Delta\,(356)} = \frac{J(123)\,.\,J(456)}{J\,(124)\,.\,J\,(356)},$$

and therefore that this function is graphometric. But the same proposition is not true in respect of lines. To find the graphometric function of three lines, we observe that it must be J multiplied by some power of Π, since P is obviously inadmissible. Now Π is of two dimensions in each of the lines, and J of one; the function must therefore be $\frac{J}{\sqrt{\Pi}}$. Expressing this in terms of the sides and angles, we find for it the values

$$\frac{(\text{area})^2}{abc}, \quad \tfrac{1}{4}a \sin B \sin C, \frac{\text{area}}{2\,\frac{a}{\sin A}},$$

$$\frac{a}{2 \sin A}\sqrt{\{\Sigma\,(\Sigma - \sin A)\,(\Sigma - \sin B)\,(\Sigma - \sin C)\}},$$

if $$2\Sigma = \sin A + \sin B + \sin C.$$

This function I call for convenience the *Projector* of the triangle, and denote by $P(ABC)$. We have, of course,

$$\frac{P(ABC).P(DEF)}{P(ABD).P(CEF)} = \frac{J(ABC).J(DEF)}{J(ABD).J(CEF)},$$

and this ratio is therefore unaltered by projection.

By means of the transformation (8), and its reciprocal

$$\phi O.J(AB\infty)^2 = J(BO\infty)^2.\phi A + 2J(BO\infty).J(AO\infty).\psi AB + J(AO\infty)^2.\phi B,$$

the following theorems may be easily proved:

$$(\alpha)\quad \begin{vmatrix} \phi AD, & \phi AE, & \phi AF, & A\infty^2 \\ \phi BD, & \phi BE, & \phi BF, & B\infty^2 \\ \phi CD, & \phi CE, & \phi CF, & C\infty^2 \\ D\infty^2, & E\infty^2, & F\infty^2, & 0 \end{vmatrix}$$

$$\equiv J(ABC).J(DEF).A\infty.B\infty.C\infty.D\infty.E\infty.F\infty \quad \text{(in points).}$$

$$(\beta)\quad \begin{vmatrix} J(AD\infty)^2, & J(AE\infty)^2, & J(AF\infty)^2, & \phi A \\ J(BD\infty)^2, & J(BE\infty)^2, & J(BF\infty)^2, & \phi B \\ J(CD\infty)^2, & J(CE\infty)^2, & J(CF\infty)^2, & \phi C \\ \phi D, & \phi E, & \phi F, & 0 \end{vmatrix}$$

$$\equiv J(BC\infty).J(CA\infty).J(AB\infty).J(EF\infty).J(FD\infty).J(DE\infty) \quad \text{(in lines).}$$

And, in the process of proving them, we obtain

$$(\gamma)\quad \begin{vmatrix} \psi(OA, OD), & \psi(OA, OE), & \psi(OA, OF), & A\infty \\ \psi(OB, OD), & \psi(OB, OE), & \psi(OB, OF), & B\infty \\ \psi(OC, OD), & \psi(OC, OE), & \psi(OC, OF), & C\infty \\ D\infty, & E\infty, & F\infty, & 0 \end{vmatrix}$$

$$\equiv J(ABC).J(DEF).O\infty^4 \text{ (in points).}$$

$$(\delta)\quad \begin{vmatrix} \psi AD, & \psi AE, & \psi AF, & \dfrac{\phi A}{J(OA\infty)} \\ \psi BD, & \psi BE, & \psi BF, & \dfrac{\phi B}{J(OB\infty)} \\ \psi CD, & \psi CE, & \psi CF, & \dfrac{\phi C}{J(OC\infty)} \\ \dfrac{\phi D}{J(OD\infty)}, & \dfrac{\phi E}{J(OE\infty)}, & \dfrac{\phi F}{J(OF\infty)}, & 0 \end{vmatrix}$$

$$\equiv \phi O^2 \frac{J(BC\infty).J(CA\infty).J(AB\infty).J(EF\infty).J(FD\infty).J(DE\infty)}{J(OA\infty).J(OB\infty).J(OC\infty).J(OD\infty).J(OE\infty).J(OF\infty)}$$

(in lines).

To these may be added

$$(\epsilon)\quad \begin{vmatrix} AL, & BL, & CL \\ AM, & BM, & CM \\ AN, & BN, & CN \end{vmatrix} \equiv J(ABC).J(LMN)$$

(in points and lines).

The interpretations of these are :

$$(\alpha)\quad \begin{vmatrix} 14^2, & 15^2, & 16^2, & 1 \\ 24^2, & 25^2, & 26^2, & 1 \\ 34^2, & 35^2, & 36^2, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} \equiv \text{area } (123) \times \text{area } (456)$$

(where 14 denotes the distance between the points 1 and 4).

$$(\beta)\quad \begin{vmatrix} \sin^2 14, & \sin^2 15, & \sin^2 16, & 1 \\ \sin^2 24, & \sin^2 25, & \sin^2 26, & 1 \\ \sin^2 34, & \sin^2 35, & \sin^2 36, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix}$$

$$\equiv \sin 23 \,.\, \sin 31 \,.\, \sin 12 \,.\, \sin 56 \,.\, \sin 64 \,.\, \sin 45$$

(where 14 denotes the angle between the lines 1 and 4).

$$(\gamma)\quad \begin{vmatrix} \cos AOD, & \cos AOE, & \cos AOF, & \dfrac{1}{OA} \\ \cos BOD, & \cos BOE, & \cos BOF, & \dfrac{1}{OB} \\ \cos COD, & \cos COE, & \cos COF, & \dfrac{1}{OC} \\ \dfrac{1}{OD}, & \dfrac{1}{OE}, & \dfrac{1}{OF}, & 0 \end{vmatrix}$$

$$\equiv \frac{\text{area } ABC}{OA \,.\, OB \,.\, OC} \cdot \frac{\text{area } DEF}{OD \,.\, OE \,.\, OF} \text{ (in points).}$$

$$(\delta)\quad \begin{vmatrix} \cos AD, & \cos AE, & \cos AF, & \frac{1}{\sin OA} \\ \cos BD, & \cos BE, & \cos BF, & \frac{1}{\sin OB} \\ \cos CD, & \cos CE, & \cos CF, & \frac{1}{\sin OC} \\ \frac{1}{\sin OD}, & \frac{1}{\sin OE}, & \frac{1}{\sin OF}, & 0 \end{vmatrix}$$

$$\equiv \frac{\sin BC \,.\, \sin CA \,.\, \sin AB}{\sin OA \,.\, \sin OB \,.\, \sin OC} \cdot \frac{\sin EF \,.\, \sin FD \,.\, \sin DE}{\sin OD \,.\, \sin OE \,.\, \sin OF}.$$

$$(\epsilon)\quad \begin{vmatrix} AL, & BL, & CL \\ AM, & BM, & CM \\ AN, & BN, & CN \end{vmatrix} \equiv \text{area } ABC \times \text{projector of } LMN,$$

where ABC are points, and LMN lines, and AL is the perpendicular from the point A on the line L.

The interpretation of the identity

$$\begin{vmatrix} AL, & BL \\ AM, & BM \end{vmatrix} \equiv J(L, M, AB) \equiv J(LM, A, B)$$

is $\begin{vmatrix} AL, & BL \\ AM, & BM \end{vmatrix} \equiv$ projector of L, M, $AB \times$ distance AB,
$\equiv$ area of LM, A, $B \times \sin LM$,

where AL is the perpendicular as before.

Circles.

9. *To find the equation to the circle whose diameter is the line joining the intersection of A and B to the intersection of C and D.*

The circle is the locus of the intersection of two lines at right angles to one another drawn through the two points. Now if $A = kB$ and $C = k'D$ are at right angles to one another, we must have

$$\psi(A - kB,\ C - k'D),$$

or
$$\psi AC - k \,.\, \psi BC - k'\psi AD + kk'\psi BD = 0,$$

and then, eliminating k, k' by the equations $A = kB$, $C = k'D$, we have for the equation to the circle

$$BD \cdot \psi AC - AD \cdot \psi BC - BC \cdot \psi AD + AC \cdot \psi BD = 0,$$

that is to say

$$\begin{vmatrix} \psi AC, & \psi AD, & A \\ \psi BC, & \psi BD, & B \\ C, & D, & 0 \end{vmatrix} = 0.$$

10. *The circles whose diameters are the diagonals of a quadrilateral have a common radical axis.*

For, denote the equation just found by $(AB, CD) = 0$; then it is easy to verify the identity

$$(AB, CD) + (AC, DB) + (AD, BC) \equiv 0,$$

which expresses the property in question.

11. If α, β, γ, δ are the *perpendiculars* from a current point on the lines A, B, C, D, the equation to the circle may evidently be written

$$\begin{vmatrix} \cos(\alpha\gamma), & \cos(\alpha\delta), & \alpha \\ \cos(\beta\gamma), & \cos(\beta\delta), & \beta \\ \gamma, & \delta, & 0 \end{vmatrix} = 0.$$

The translation of these results into the ordinary systems of coordinates requires the determination of the Absolute, to which I now proceed.

IV. *Determination of the absolute.*

12. As the title of this section might be productive of mixed feelings, I will say at once that "to determine the Absolute" means "to find the form of ϕ." It is not pretended that the method here used is simpler than that generally given; but only that it is more suggestive and of wider application.

Trilinears.

The trilinear coordinates of a point are three quantities proportional to the perpendicular distances of the point from three given lines, $X=0$, $Y=0$, $Z=0$. We may therefore write, if x, y, z are these coordinates,

$$x,\ y,\ z = \frac{X}{\sqrt{\phi X}},\quad \frac{Y}{\sqrt{\phi Y}},\quad \frac{Z}{\sqrt{\phi Z}},$$

and so

$$\begin{aligned}\phi(lx+my+nz)&\\ &= l^2\phi x + m^2\phi y + n^2\phi z + 2mn\psi yz + 2nl\psi zx + 2lm\psi xy\\ &= l^2 + m^2 + n^2 + 2mn\cos(YZ) + 2nl\cos(ZX)\\ &\qquad + 2lm\cos(XY).\end{aligned}$$

It only remains to determine *which* cosine is meant in each case. To do this we make the convention that x, y, z shall be all positive for a point inside the triangle. We can then see geometrically, that a line perpendicular to BC (or X) through B (or ZX), is represented by

$$z + x\cos B = 0,$$

therefore

$$\psi(x + z + x\cos B) \equiv \psi(x, z) + \cos B = 0,$$
$$\text{or } \cos(XZ) = -\cos B.$$

Hence, by symmetry, we have,

$$\begin{aligned}\phi(lx+my+nz) &= l^2 + m^2 + n^2\\ &\qquad - 2mn\cos A - 2nl\cos B - 2lm\cos C,\end{aligned}$$

where A, B, C are the internal angles of the triangle.

One may now safely see, *a priori*, that the angle between two straight lines *ought* to be the angle through which one of them must be turned in order that its positive side may coincide with the positive side of the other. And by aid of this definition the result we have just found may be extended from Trilinear to Multilinear systems. We shall always have

$$\phi(lx + my + nz + \ldots) \equiv l^2 + m^2 + n^2 \ldots - 2lm\cos(xy) - \ldots$$

Areals.

The Areal coordinates of a point are quantities proportional to the triangles it determines with the sides of a certain fixed triangle. Let α, β, γ be the sides of this triangle; then we have

$$x,\ y,\ z = \frac{\alpha X}{\sqrt{\phi X}},\ \frac{\beta Y}{\sqrt{\phi Y}},\ \frac{\gamma Z}{\sqrt{\phi Z}},$$

and so

$$\phi\,(lx+my+nz)$$
$$= \alpha^2 l^2 + \beta^2 m^2 + \gamma^2 n^2 - 2\beta\gamma mn\cos A - 2\gamma\alpha nl\cos B - 2\alpha\beta lm\cos C,$$

the signs being determined as before.

Interpretation of Constants.

13. Let P_1, P_2, P_3 be the three perpendiculars of the triangle of reference, and ϖ_1, ϖ_2, ϖ_3 the perpendiculars from the angular points on the line (lmn). Then by the formula in Art. 7, we have, in Trilinears,

$$\varpi_1 : \varpi_2 : \varpi_3 = lP_1 : mP_2 : nP_3,$$

since, in this system, the coordinates of the angular points are as

$$P_1,\ 0,\ 0;\quad 0,\ P_2,\ 0;\quad 0,\ 0,\ P_3.$$

Consequently,

$$l : m : n = \frac{\varpi_1}{P_1} : \frac{\varpi_2}{P_2} : \frac{\varpi_3}{P_3}.$$

In the Areal system, the coordinates of the angular points are as 1, 0, 0; 0, 1, 0; 0, 0, 1; so that in this case we have

$$l : m : n = \varpi_1 : \varpi_2 : \varpi_3.$$

It is of the greatest importance *to notice* that there are *two different* systems of Tangential Coordinates, corresponding to Trilinears and Areals respectively. In the former, the coordinates of a line are proportional to $\frac{\varpi_1}{P_1}$, $\frac{\varpi_2}{P_2}$, $\frac{\varpi_3}{P_3}$; in the latter they are proportional to ϖ_1, ϖ_2, ϖ_3. If we keep this

distinction in mind, we may say generally that the *coefficients* in the *equation* of a point are proportional to the *coordinates* of the point; and that the *coefficients* in the *equation* of a line are proportional to the *coordinates* of the line. And then we may get rid of coordinates altogether, and consider nothing but equations.

Line at Infinity.

14. Form the reciprocal of ϕ by the ordinary method; it is, in Trilinears,

$$(x \sin A + y \sin B + z \sin C)^2,$$

and in Areals,

$$\beta^2\gamma^2 \sin^2 A\,(x+y+z)^2.$$

It must be remembered that we are not here finding the *equation* of the line at infinity, but the value of the function $A\infty^2$ of the point (x, y, z). In the case of Trilinears, if we suppose x, y, z to be actually *equal* to the perpendiculars from a point on these sides, then the quantity

$$x \sin A + y \sin B + z \sin C$$

is four times the projector of the triangle of reference.

Quadrilaterals.

15. Consider four straight lines $A, B, C, D = 0$. Since the equation of *any* line can be expressed in terms of the equations of three others, there must be an identical relation

$$lA + mB + nC + sD \equiv 0 \ldots\ldots\ldots\ldots\ldots (1),$$

and because this is independent of the values of x, y, z, we must have

$$\begin{aligned} lA_x + mB_x + nC_x + sD_x &= 0, \\ lA_y + mB_y + nC_y + sD_y &= 0, \\ lA_z + mB_z + nC_z + sD_z &= 0, \end{aligned}$$

where A_x denotes the coefficient of x in A, &c. Eliminating therefore l, m, n, s from these four equations, we have

$$\begin{vmatrix} A, & B, & C, & D \\ A_x, & B_x, & C_x, & D_x \\ A_y, & B_y, & C_y, & D_y \\ A_z, & B_z, & C_z, & D_z \end{vmatrix} \equiv 0.$$

But, by definition of a Jacobian, this is equivalent to

$$A\,.\,J(BCD) - B\,.\,J(CDA) + C\,.\,J(DAB) - D\,.\,J(ABC) \equiv 0,$$

and this therefore is the identical relation between four given lines.

Now it is often convenient to use a set of four coordinates, x, y, z, w, connected by the identical relation

$$x + y + z + w = 0\,;$$

we have then only to put

$$x, y, z, w = A\,.\,J(BCD),\quad B\,.\,J(CAD),\quad C\,.\,J(ABD),\quad D\,.\,J(CBA),$$

and we shall have such a system. We then find that

$$\phi(lx + my + nz + sw)$$
$$= l^2\phi A\,.\,J(BCD)^2 + \ldots + 2lm\,.\,\psi AB\,.\,J(BCD)\,.\,J(CAD) + \ldots$$

Now let $\alpha, \beta, \gamma, \delta$ be the projectors of the triangles BCD, CAD, ABD, CBA; and divide the result just found by $\phi A\,.\,\phi B\,.\,\phi C\,.\,\phi D$; then we may write

$$\phi(lx + my + nz + sw) = \alpha^2 l^2 + \beta^2 m^2 + \gamma^2 n^2 + \delta^2 s^2 - 2\alpha\beta lm\cos(xy) - \ldots\ldots$$

We have here in fact made x proportional to the perpendicular from a point on A, multiplied by the projector of BCD. And this might be taken as the definition of the system of coordinates.

If for D we write ∞, the identity becomes

$$\infty\,.\,J(ABC) \equiv A\,.\,J(BC\infty) + B\,.\,J(CA\infty) + C\,.\,J(AB\infty).$$

Hence we find, in Trilinears, for instance,

$$\infty\,.\,\frac{J(ABC)}{\sqrt{\phi A\,.\,\phi B\,.\,\phi C}} \equiv x\sin A + y\sin B + z\sin C \equiv P(ABC),$$

which agrees with the result before obtained, since

$$\frac{J(ABC)}{\sqrt{\phi A\,.\,\phi B\,.\,\phi C}}$$

is really the *ratio* of the projector of ABC to the projector of the triangle of reference.

Equations.

16. The *condition* that a point A shall lie on a line L is $AL=0$. Now if we consider the coefficients of the point as coordinates, then $AL=0$ is the *equation* of the *line* L; if we consider the coefficients of the line as coordinates, $AL=0$ is the equation of the *point* A. It is not difficult to see how this notion may be generalized. Consider the form

$$\xi x+\eta y+\zeta z,$$

which may represent either a point or a line, according as $(\xi\eta\zeta)$ or (xyz) are regarded as variable. I shall denote this form by an asterisk (*), and use it exclusively to represent *equations*. For instance, the equation $J(AB^*)=0$ is the equation to the *point* or *line* AB, according as A and B are lines or points. I give one or two examples of equations found by this method.

(α) *Locus of a point subtended by four given points in a given anharmonic ratio.*

We want a point P such that $\dfrac{\sin APB\,.\sin CPD}{\sin APC\,.\sin DPB}=k$. The locus is then immediately written down; viz. since

$$\sin APB=\frac{P\infty\,.J(ABP)}{\sqrt{\phi}\,(AP)\,.\,\phi\,(BP)},$$

it is

$$J(AB^*)\,.\,J(CD^*)=kJ(AC^*)\,.\,J(DB^*),$$

which is clearly a conic through the points $ABCD$.

(β) *Envelop of a line cut by four given lines in a given anharmonic ratio.*

By the formula, Art. 7, the envelop is (in Tangential Coordinates),

$$J(LM^*)\,.\,J(NR^*)=k\,.\,J(LN^*)\,.\,J(RM^*),$$

a conic touching the four given lines.

(γ) *Envelop of a line, the product of whose distances from two fixed points is constant.*

Ans. $(A^*)\,(B^*) = k^2 A\infty \,.\, B\infty \,.\, \phi\,(^*)$,

a conic inscribed in the quadrilateral formed by joining A and B to the two points of ϕ.

(δ) *Envelop of a line of constant length resting on two given lines.*

Ans. $\phi\,(^*)\,.\,J\,(LM^*)^2 = k^2\,.\,J\,(L\infty\,^*)^2\,.\,J\,(M\infty\,^*)$.

Let α be the point LM, β the point $L\infty$, γ the point $M\infty$; then if we remember that any point on the line $\alpha\rho$, *i.e.* at infinity, has its equation of the form $l\alpha + m\beta = 0$, this equation may be written

$$\alpha^2\,(a\beta^2 + 2b\beta\gamma + c\gamma^2) = k^2\beta^2\gamma^2,$$

or

$$\frac{a}{\gamma^2} + \frac{2b}{\beta\gamma} + \frac{c}{\beta^2} = \frac{k^2}{\alpha^2};$$

the envelop is therefore the tangential inverse, in respect of the triangle $\alpha\beta\gamma$, of a conic in respect of which α is the pole of $\beta\gamma$. It is obvious from the figure that each of the lines L, M, touches at two cusps, so that the curve is of the sixth order. The equation shews that the line at infinity ($\beta\gamma$) is also a double tangent, and when L, M are at right angles, $b = 0$, and the line at infinity touches at two cusps.

(ϵ) *Envelop of a line cut by three given lines in given ratios.*

Ans. $\lambda J\,(MN^*)\,.\,J\,(L\infty\,^*) + \mu J\,(NL^*)\,.\,J\,(M\infty\,^*) + \nu J\,(LM^*)\,J\,(N\infty\,^*) = 0,$

a parabola touching the three given lines.

The utility of the method in questions of this sort is still more evident in the case of curves and surfaces of the second order.

V. *Planes and Points in Space.*

Expressions Considered.

17. In Geometry of Three Dimensions the absolute is an imaginary circle in which the plane at infinity cuts any sphere. The condition that a plane $A = 0$ may touch this circle, is of the second order in the coefficients of A, and may be written $\phi(A) = 0$. This being so, we have, as before,

$$\phi(\lambda A + \mu B) \equiv \lambda^2 \phi(A) + 2\lambda\mu \,.\, \psi(A, B) + \mu^2 \,.\, \phi(B),$$

where the function $\psi(A, B)$ is of the first order in the coefficients of each of the two planes, and vanishes only when they are at right angles.

The condition that a point $a = 0$ may be at infinity is of the first order in the coordinates or coefficients of the point, and may be written $a\,\infty = 0$.

The condition that four planes $A, B, C, D = 0$ may meet in a point, is that the determinant formed with their coordinates or coefficients, that is to say, their Jacobian, shall vanish. This I express by the equation

$$J(ABCD) = 0.$$

The condition that three planes $A, B, C = 0$ may meet at infinity, or, which is the same thing, may be parallel to the same line, is accordingly

$$J(ABC\,\infty) = 0.$$

18. The form of the absolute ϕ is found exactly as in plane geometry; I will therefore anticipate a formula, and give it here. We have generally,

$$\phi(lx + my + nz + sw) \equiv l^2\phi(x) + \ldots + 2mn\psi(y, z) + \ldots .$$

Now by means of the formula

$$\cos(A, B) = \frac{\psi(A, B)}{\sqrt{\phi(A) \,.\, \phi(B)}},$$

this becomes, in quadriplanar coordinates,

$$\phi\,(lx+my+nz+sw) \equiv l^2+m^2+n^2+s^2-2mn\cos(y,z)-\ldots,$$

and in tetrahedral coordinates

$$\phi\,(lx+my+nz+sw) \equiv l^2\alpha^2+\ldots-2mn\,.\,\beta\gamma\,.\cos(y,z)-\ldots,$$

in both of which $\cos(y,z)$ means the cosine of the internal angle between the planes $y=0$, $z=0$ of the fundamental tetrahedron, and α, β, γ, δ are the areas of the faces. The equation to the plane at infinity is then found to be, in the quadriplanar system,

$$Px+Qy+Rz+Sw=0,$$

where
$$P^2=\begin{vmatrix} 1, & -\cos(y,z), & -\cos(y,w) \\ -\cos(y,z), & 1, & -\cos(z,w) \\ -\cos(y,w), & -\cos(z,w), & 1 \end{vmatrix}.$$

It is convenient to call P the *sine* of the solid angle yzw (M. Paul Serret). If we write A, B, C, D for the solid angles of the tetrahedron, then

$$\frac{\sin A}{\alpha}=\frac{\sin B}{\beta}=\frac{\sin C}{\gamma}=\frac{\sin D}{\delta},$$

and the equation to the plane at infinity, in tetrahedral coordinates, is

$$\beta\gamma\delta\sin A\,(x+y+z+w)=0.$$

Fundamental Propositions.

19. The following propositions may be proved by Cartesian coordinates.

A. If a plane touch the imaginary circle at infinity,

(*a*) Every area measured on the plane is zero.

(*b*) Every perpendicular distance from it is infinite.

(*c*) Of every angle *on* the plane, the sine is zero and the cosine unity.

(*d*) Of every angle made with the plane, the sine and cosine are infinite.

B. If a straight line meet the imaginary circle at infinity,

(*a*) Every length measured on the line is zero.

(*b*) Every perpendicular distance from it is infinite.

(*c*) Of the angle between any two planes through it, the sine is zero and the cosine unity.

(*d*) Of every angle made with it, the sine and cosine are infinite.

C. If a point be at infinity,

(*a*) Its perpendicular distance from any plane or straight line not passing through it is infinite.

(*b*) The volume of the tetrahedron which it forms with any other three points not in the same plane with it is infinite.

(*c*) The area of the triangle which it forms with any other two points, not such that the plane of the triangle touches the absolute (see prop. *A*, *a*), is infinite.

(*d*) Its distance from any other point, not at infinity, is infinite.

Formulæ of Adaptation.

20. These are to be deduced from the propositions just stated precisely in the same way as the corresponding formulæ in Plane Geometry were proved. I postpone for the present the consideration of formulæ relating to straight lines.

The volume contained by four points a, b, c, d is

$$\frac{J(abcd)}{a\infty \,.\, b\infty \,.\, c\infty \,.\, d\infty}.$$

If the points are given as the intersections of four planes, A, B, C, D, the expression becomes

$$\frac{\{J(ABCD)\}^3}{J(BCD\infty)\,.\,J(CDA\infty)\,.\,J(DAB\infty)\,.\,J(ABC\infty)}.$$

The area contained by three points a, b, c is

$$\frac{\sqrt{\phi}\,(abc)}{a\varpi \,.\, b\varpi \,.\, c\varpi},$$

where $\phi\,(abc)$ means that we are to write down the equation of the plane through a, b, c, and then form the condition that it may touch the imaginary circle at infinity. If the three points are given as intersections of the plane A with the planes B, C, D, the formula becomes

$$\frac{\sqrt{\phi A}\,.\,\{J\,(ABCD)\}^2}{J(ACD\,\infty)\,.\,J\,(ADB\,\infty)\,.\,J\,(ABC\,\infty)},$$

which is the area determined on the plane A by the planes B, C, D.

The perpendicular from the point a on the plane A is $\frac{aA}{\sqrt{\phi A}\,.\,a\,\infty}$. Here aA is used for the result of substituting the coordinates of a in the equation of A, or *vice versâ*.

If θ be the angle between two planes A and B, then

$$\cos\theta = \frac{\psi\,(A,\,B)}{\sqrt{\phi A}\,.\,\phi B}, \quad \sin^2\theta = \frac{\phi A\,.\,\phi B - \{\psi\,(A,\,B)\}^2}{\phi A\,.\,\phi B}.$$

It will be proved that the sine of the solid angle contained by three planes is

$$\frac{J\,(ABC\,\infty)}{\sqrt{\phi A\,.\,\phi B\,.\,\phi C}}.$$

Theorems.

21. I use Professor Sylvester's umbral notation, which, whenever determinants have to be employed, is not only convenient, but essential.

In this notation, the determinant

$$\begin{vmatrix} \psi\,(AD), & \psi\,(BD), & \psi\,(CD) \\ \psi\,(AE), & \psi\,(BE), & \psi\,(CE) \\ \psi\,(AF), & \psi\,(BF), & \psi\,(CF) \end{vmatrix}$$

is written

$$\psi \begin{vmatrix} ABC \\ DEF \end{vmatrix}.$$

It will be easy now to interpret the notation in other cases.

For instance, the definition of $\sin(y, z, w)$, in Art. 18, may be written

$$\sin^2(y, z, w) \equiv \cos \begin{vmatrix} yzw \\ yzw \end{vmatrix}.$$

I consider now the series of determinants

$$\psi \begin{vmatrix} ABCD \\ EFGH \end{vmatrix}, \quad \psi \begin{vmatrix} ABC \\ DEF \end{vmatrix}, \quad \psi \begin{vmatrix} AB \\ CD \end{vmatrix}.$$

Suppose we wanted to find the condition that it might be possible to draw a plane

$$\lambda A + \mu B + \nu C + \sigma D = 0,$$

which should be at once perpendicular to each of the planes E, F, G, H. We should write down four equations like

$$\lambda\psi(A, E) + \mu\psi(B, E) + \nu\psi(C, E) + \sigma\psi(D, E) = 0,$$

and then eliminate $\lambda\mu\nu\sigma$ between these equations. Thus, we should arrive at the condition $\psi \begin{vmatrix} ABCD \\ EFGH \end{vmatrix} = 0$. First, suppose that A, B, C, D do not meet in a point. Then the equation of any plane whatever can be put into the form

$$\lambda A + \mu B + \nu C + \sigma D = 0.$$

Now, the plane at infinity is perpendicular to all other planes. In this case, therefore, the condition $\psi \begin{vmatrix} ABCD \\ EFGH \end{vmatrix} = 0$ is satisfied. Next let A, B, C, D meet in a point. Then there is an *identity*

$$\lambda A + \mu B + \nu C + \sigma D \equiv 0,$$

which gives rise to four other identities like

$$\lambda\psi(AE) + \mu\psi(BE) + \nu\psi(CE) + \sigma\psi(DE) \equiv 0,$$

so that in this case also $\psi \begin{vmatrix} ABCD \\ EFGH \end{vmatrix} = 0$.

Hence, we have always, identically,

$$\psi \begin{vmatrix} ABCD \\ EFGH \end{vmatrix} \equiv 0 \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

In the same way we see, that $\psi \begin{vmatrix} ABC \\ DEF \end{vmatrix} = 0$, expresses the condition that a plane $\lambda A + \mu B + \nu C = 0$ may be so drawn as to

be perpendicular at once to each of the planes D, E, F. Now unless three planes are parallel to the same line, the only plane which is perpendicular to all three of them is the plane at infinity. If therefore the condition is satisfied, either A, B, C meet at infinity, or D, E, F meet at infinity. Thus we have

$$\psi \begin{vmatrix} ABC \\ DEF \end{vmatrix} \equiv J(ABC\infty) \,.\, J(DEF\infty) \ldots\ldots\ldots (2).$$

The condition $\psi \begin{vmatrix} AB \\ CD \end{vmatrix} \doteq 0$ will be satisfied if we can draw a plane through the line (A, B) perpendicular to the two planes C, D. It is easy to see that this can only be the case when the line (A, B) is perpendicular to the line (C, D). This result may be expressed in the form

$$\psi \begin{vmatrix} AB \\ CD \end{vmatrix} \equiv \psi(AB,\ CD) \ldots\ldots\ldots\ldots (3).$$

To interpret the theorems (1) and (2), I observe that to multiply any single row or column of a determinant by a certain quantity, is to multiply the whole determinant by that quantity. Thus we have

$$\cos \begin{vmatrix} ABCD \\ EFGH \end{vmatrix} = \frac{\psi \begin{vmatrix} ABCD \\ EFGH \end{vmatrix}}{\sqrt{\phi A \,.\, \phi B \,.\, \phi C \,.\, \phi D \,.\, \phi E \,.\, \phi F \,.\, \phi G \,.\, \phi H}} = 0,$$

$$\text{and } \cos \begin{vmatrix} ABC \\ DEF \end{vmatrix} = \frac{\psi \begin{vmatrix} ABC \\ DEF \end{vmatrix}}{\sqrt{\phi A \,.\, \phi B \,.\, \phi C \,.\, \phi D \,.\, \phi E \,.\, \phi F}}$$

$$= \frac{J(ABC\infty)}{\sqrt{\phi A \,.\, \phi B \,.\, \phi C}} \cdot \frac{J(DEF\infty)}{\sqrt{\phi D \,.\, \phi E \,.\, \phi F}}.$$

If now we make A, B, C identical with D, E, F respectively, we get

$$\frac{\{J(ABC\infty)\}^2}{\phi A \,.\, \phi B \,.\, \phi C} = \cos \begin{vmatrix} ABC \\ ABC \end{vmatrix} = \sin^2(A, B, C) \text{ by definition; and}$$

$$\text{so} \qquad \cos \begin{vmatrix} ABC \\ DEF \end{vmatrix} = \sin(A, B, C) \,.\, \sin(D, E, F).$$

The theorems (1) and (2) will be found to embody a great number of results.

22. I now prove certain known theorems of determinants, which are useful in this subject.

Consider five planes A, B, C, D, E, whose equations are

$$l_1x + m_1y + n_1z + s_1w = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$l_5x + m_5y + n_5z + s_5w = 0.$$

We know that, since each one can be expressed in terms of the other four, there must exist some identical relation

$$PA + QB + RC + SD + TE = 0.$$

And because this is true for all values of $x_1y_1z_1w_1$, we must have

$$Pl_1 + Ql_2 + Rl_3 + Sl_4 + Tl_5 = 0,$$
$$\text{\&c.} \qquad\qquad \text{\&c.}$$

From these five equations we can eliminate $PQRST$, and we find

$$A\,.\,J(BCDE) + B\,.\,J(CDEA) + C\,.\,J(DEAB) + D\,.\,J(EABC) + E\,.\,J(ABCD) \equiv 0 \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

the identical relation in question.

Now if we substitute in A the coordinates of the intersection of F, G, H, the result is clearly $J(AFGH)$. But the equation (4) is true for all values of the variables; we may therefore substitute in it the coordinates of the point (F, G, H). In this way we obtain

$$J(AFGH)\,.\,J(BCDE) + J(BFGH)\,.\,J(CDEA) + J(CFGH)\,.\,J(DEAB) + J(DFGH)\,.\,J(EABC) + J(EFGH)\,.\,J(ABCD) \equiv 0 \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Making H identical with E, and transposing, we have

$$J(ABCE)\,.\,J(DFGE) \equiv J(AFGE)\,.\,J(BCDE) + J(BFGE)\,.\,J(CADE) + J(CFGE)\,.\,J(ABDE) \ldots\ldots (6).$$

Write ∞ for E, and use theorem (2); thus

$$\cos \begin{vmatrix} ABC \\ DFG \end{vmatrix} \equiv \cos \begin{vmatrix} AFG \\ BCD \end{vmatrix} + \cos \begin{vmatrix} BFG \\ CAD \end{vmatrix} + \cos \begin{vmatrix} CFG \\ ABD \end{vmatrix} \ldots\ldots (7).$$

Again, let E be the plane $w=0$ of the fundamental tetrahedron; and write (1, 2, 3, 4, 5, 6) for the traces on this plane of the planes (A, B, C, D, F, G) respectively; then we get the theorem in plane geometry

$$J(123).J(456) \equiv J(156).J(234) + J(164).J(235) + J(145).J(236).$$

If we make the lines 1, 2 identical, and write ∞ for each, this becomes the theorem referred to in Art. 6.

Spheres.

23. *To find the equation to the sphere whose diameter is the line joining the point* (A, B, C), *to the point* (D, E, F).

The sphere may be defined as the locus of the foot of a perpendicular from the point (A, B, C) on a variable plane passing through the point (D, E, F). Now the equations to a line through (A, B, C) perpendicular to a plane L are

$$\frac{A}{\psi(A, L)} = \frac{B}{\psi(B, L)} = \frac{C}{\psi(C, L)} = \frac{1}{k}, \text{ suppose.}$$

Let now $L \equiv \lambda D + \mu E + \nu F$; then at the foot of the perpendicular,

$$\begin{aligned}
\lambda\psi(A, D) + \mu\psi(A, E) + \nu\psi(A, F) - kA &= 0,\\
\lambda\psi(B, D) + \mu\psi(B, E) + \nu\psi(B, F) - kB &= 0,\\
\lambda\psi(C, D) + \mu\psi(C, E) + \nu\psi(C, F) - kC &= 0,\\
\lambda D + \mu E + \nu F &= 0,
\end{aligned}$$

from these equations we can eliminate λ, μ, ν, k, and we get for the equation of the sphere

$$\begin{vmatrix}
\psi AD, & \psi AE, & \psi AF, & A\\
\psi BD, & \psi BE, & \psi BF, & B\\
\psi CD, & \psi CE, & \psi CF, & C\\
D & E & F & 0
\end{vmatrix} = 0.$$

Call this $(\psi)\begin{vmatrix} DEF0 \\ ABC0 \end{vmatrix} = 0$; then since theorem (6) is a general theorem of determinants, we have,

$$(\psi)\begin{vmatrix} ABC0 \\ DEF0 \end{vmatrix} \equiv (\psi)\begin{vmatrix} AEF0 \\ BCD0 \end{vmatrix} + (\psi)\begin{vmatrix} AFD0 \\ BCE0 \end{vmatrix} + (\psi)\begin{vmatrix} ADE0 \\ BCF0 \end{vmatrix}; \quad \ldots(8),$$

these four spheres have therefore a common radical axis. Whence this geometrical theorem:

If the faces of a tetrahedron A, B, C, D are met by a straight line in the points a, b, c, d; then the spheres whose diameters are Aa, Bb, Cc, Dd have a common radical axis.

It will be observed that the *faces* of the tetrahedron are A, D, E, $F = 0$, and that the *straight line* is the intersection of the planes $B = 0$, $C = 0$. The relation (8) is very easily verified.

*(To be continued.)**

* [No more on this subject was published in the *Quarterly Journal.*]

XII.

ON THE GENERAL THEORY OF ANHARMONICS*.

1. THE theory of Anharmonics on the straight line may be stated in the following symmetrical form :—

(i) There is an identical relation connecting the distances of four points, 1, 2, 3, 4, on a right line, viz.,

$$12 . 34 + 13 . 42 + 14 . 23 \equiv 0.$$

(ii) The ratios of the three terms in this identity are not altered by projection or integral linear transformation. There is a corresponding theory of four straight lines meeting in a point.

2. In applying this theory to geometry of two dimensions, we meet with this third proposition :—

(iii) If a straight line meet four fixed straight lines, so that the distances of the points which they determine on it satisfy a relation of the form

$$\lambda . 12 . 34 + \mu . 13 . 42 + \nu . 14 . 23 = 0,$$

then the envelope of the line is of the second class, touching the four given lines. There is, of course, a correlative proposition on the locus of a point subtended by four points in a given manner. The propositions (i), (ii), (iii), each including its converse and correlative propositions, may be regarded as constituting the entire theory of anharmonics in geometry of one dimension, and its application to geometry of two dimensions.

* [From the *Proceedings of the London Mathematical Society*, Vol. II. No. 9, pp. 3—6.]

3. I proceed to state the theory of anharmonics in geometry of two dimensions.

(i) There are six identical relations connecting the areas of triangles formed by six points, 1, 2, 3, 4, 5, 6, in a plane, viz.,

$$123 . 456 + 124 . 563 + 125 . 634 + 126 . 345 \equiv 0,$$

with five others obtained from this by permutation.

(ii) The ratios of the terms in these identities are not altered by projection or integral linear transformation. Under each of these propositions are included *three* correlatives, to explain which I must introduce three new definitions:

DEF. 1st. The *projector* of a plane triangle is the square of the area divided by the continued product of the sides.

DEF. 2nd. In a solid angle considered as determined by three concurrent straight lines, the sine of the angle between the first line and the plane of the other two, multiplied by the sine of the angle between the other two lines, is a symmetrical quantity in respect of the three lines, and is called the *sine* of the three lines.

DEF. 3rd. In a solid angle considered as determined by three planes, the sine of the angle between the first plane and the intersection of the other two, multiplied by the sine of the angle between the other two planes, is a symmetrical quantity in respect of the three planes, and is called the *sine* of the three planes.

4. It is convenient for several purposes to use the word "distance" as including all these notions: thus I shall speak of the *distance* of two lines in a plane, meaning the sine of the angle between them;

the distance of three points is the area of their triangle;

" " of three straight lines in a plane, the projector of their triangle;

" " of three planes, the sine of the planes;

" " of three concurrent lines in space, the sine of the lines.

And I shall have occasion afterwards to define the distance of four points and of four planes in space. By means of these definitions, the propositions (i), (ii) may be interpreted in four different ways, corresponding to the four aspects of bi-dimensional extension: the symbol 123 being always understood to mean the *distance* of the three things considered.

5. The propositions (i), (ii), including their converse and correlative propositions, constitute the entire theory of anharmonics in geometry of two dimensions. To apply this theory to geometry of three dimensions, I state the following proposition, which has only *one* correlative:

(iii) If a plane meet six fixed planes so that the distances of the lines they determine upon it satisfy a relation of the form

$$\lambda . 123 . 456 + \mu . 124 . 563 + \nu . 156 . 234 + \ldots = 0,$$

then the envelope of the plane is of the second class, touching the six given planes.

6. The theory of anharmonics in three dimensions is so entirely analogous to the two former theories, that it wants no further discussion. The distance of four points is the volume of their tetrahedron, and the distance of four planes is the cube of the volume divided by the product of the areas of the faces.

7. Two finite lines $11'$, $22'$, measured on the same straight line, are said to be *harmonically situate* when

$$12 . 1'2' + 12' . 1'2 = 0;$$

and when this is the case, if the two pairs of points be represented by the equations $U=0$, $V=0$, there is an invariant relation connecting the quadrics U, V which may be denoted by

$$\square (U, V) = 0;$$

the notation indicating that if we expand the discriminant of $\lambda U + \mu V$, or

$$\square (\lambda U + \mu V),$$

then the coefficient of $\lambda\mu$ in the expansion vanishes.

There is a similar relation between two angles having a common vertex; when this relation holds, I say that the two covertical angles are harmonically situate. I also speak of an harmonic pair of angles, or of an harmonic pair of finite lines, or lengths; meaning in this case covertical angles, or collinear lengths.

8. This being so, it is known that two lengths (I use the word *length* to denote a pair of points), anyhow placed in a plane, determine a conic passing through their ends, which conic is the locus of points which the lengths subtend harmonically, that is, in a pair of harmonic angles. I call this the harmonic conic of the two lengths. This (with the correlative propositions) completes the theory of harmonics in one dimension, and its application to two dimensions. I now come to consider harmonics in two dimensions.

9. If the harmonic conic of two lengths $11'$, $22'$ divides harmonically a third length $33'$, then the relation between the three lengths is symmetrical, and I say that the three lengths are *harmonically situate* in the plane. The following relation subsists among their distances, viz.,

$$123 \,.\, 1'2'3' + 1'23 \,.\, 12'3' + 12'3 \,.\, 1'23' + 123',\ 1'2'3 = 0.$$

And if each length, or point-pair, be considered as a degenerate conic, so that the equations to the three point-pairs are U, V, $W = 0$, then, when the three lengths are harmonically situate, there is an invariant relation connecting the quadrics U, V, W, which may be denoted by

$$\square\,(U, V, W) = 0;$$

$\square\,(U, V, W)$ denoting the coefficient of $\lambda\mu\nu$ in the expansion of

Discriminant of $(\lambda U + \mu V + \nu W)$.

It is obvious that a similar relation may subsist among three angles in a plane, three pairs of lines through a point in space, or three pairs of planes through a point in space.

10. Three lengths anyhow placed in space determine a quadric surface passing through their ends, which surface is the locus of points which the lengths subtend harmonically, that is,

in a triad of harmonic angles. I call this the harmonic quadric of the three lengths.

11. If the harmonic quadric of three lengths 11′, 22′, 33′ divides harmonically a fourth length 44′, then the relation between the four lengths is symmetrical, and I say that the four lengths are *harmonically situate* in space. The following relation subsists among their distances, viz.,

$$\Sigma . 1234 . 1'2'3'4' = 0 \text{ (eight terms).}$$

And if each length, or point-pair, be considered as a degenerate quadric, so that the equations to the four point-pairs are $U = 0$, $V = 0$, $W = 0$, $T = 0$, then, when the four lengths are harmonically situate, there is an invariant relation connecting the quadrics U, V, W, T, which may be denoted by

$$\square (U, V, W, T) = 0;$$

$\square (U, V, W, T)$ denoting the coefficient of $\lambda\mu\nu\rho$ in the expansion of

$$\text{Discriminant of } (\lambda U + \mu V + \nu W + \rho T).$$

It is obvious that a similar relation may subsist among four pairs of planes.

12. It only remains to explain the meaning of the conditions

$$\square (U, V, W) = 0,$$
$$\square (U, V, W, T) = 0,$$

when the quadrics do *not* break up into factors. Two conics U, V determine an harmonic conic F, locus of points which they subtend in a pair of harmonic angles. If a triangle self-conjugate of F can be inscribed in W, then the relation between UVW is symmetrical, and $\square (U, V, W) = 0$. The conics may then be spoken of as three mutually harmonic conics; a similar relation may hold between three covertical cones. Thus, in fact, three quadric *surfaces* U, V, W determine an harmonic quadric F, locus of points which they subtend in three harmonic cones. If a tetrahedron self-conjugate of F can be inscribed in T, then the relation between U, V, W, T is symmetrical, and $\square (U, V, W, T) = 0$.

XIII.

ON A GENERALIZATION OF THE THEORY OF POLARS*.

(THE present Note establishes the idea of the polar curve of a curve of given class in respect of a curve of given order, the class being less than the order; and of the polar curve of a curve of given order in respect of a curve of given class, the order being less than the class. It also deals with a certain invariant of two curves, such that the order of one is equal to the class of the other; and with certain other invariants and contravariants arising out of the theory of polars. I desire to present these ideas by themselves to the Society, because they seem likely to be useful for other purposes than that to which I propose to apply them subsequently, viz., the extension of Grassmann's Geometric Analysis.)

1. Let B_n be a curve of the nth order, and c_m a curve of the mth class. Let the equations of the curves be

$$B_n \equiv (A, B, C, D \ldots\ldots \Sigma x\, y\, z)^n$$
$$c_m \equiv (a, b, c, d \ldots\ldots\ldots \Sigma \xi\, \eta\, \zeta)^m$$

in point and line coordinates respectively.

In c_m write $\frac{\delta}{dx}, \frac{\delta}{dy}, \frac{\delta}{dz}$ in place of ξ, η, ζ respectively, and operate on B_n with the symbol thus formed. I denote the result by merely writing c_m as an operator before B_n; thus

$$c_m B_n \equiv \left(a, b, c \ldots\ldots \Sigma \frac{\delta}{dx}, \frac{\delta}{dy}, \frac{\delta}{dz}\right)^m . (A, B, C, \ldots\ldots \Sigma x, y, z)^n;$$

* [From the *Proceedings of the London Mathematical Society*, Vol. II. No. 16, pp. 116—118.]

then we find

(i) If m is less than n, $c_m B_n$ is a covariant, which I call the *polar curve* of c_m in respect of B_n. It is given in the point coordinates x, y, z, and is of order $n-m$.

(ii) If m is equal to n, $c_n B_n$ is an invariant of the two curves. When this invariant vanishes, I say that the curves are *harmonic* of each other.

(iii) If m is greater than n, $c_m B_n = 0$ always.

2. It is important to shew that these new meanings of the words *polar* and *harmonic* include the old meanings. Now the mth polar of a point (x, y, z) whose tangential equation is $x\xi + y\eta + z\zeta = 0$, say the point P, is

$$\left(x\frac{\delta}{dx} + y\frac{\delta}{dy} + z\frac{\delta}{dz}\right)^m . B_n = 0;$$

which will be denoted in our present notation by $p^m B_n = 0$. But this is, according to the new definition, the polar of m times the point p; the point p taken m times being of course a particular case of a curve of the mth class. Again, the two conics

$$(a, b, c, f, g, h)(\xi\ \eta\ \zeta)^2, \qquad (A, B, C, F, G, H)(x, y, z)^2$$

have been called *harmonic* when

$$Aa + Bb + Cc + 2Ff + 2Gg + 2Hh = 0,$$

the invariant being obviously the result of turning the first into an operator and applying it to the second.

3. Returning to the curves c_m, B_n, we may convert B_n into an operator by writing in it $\frac{\delta}{d\xi}, \frac{\delta}{d\eta}, \frac{\delta}{d\zeta}$ in place of x, y, z respectively. The result of the operation on c_m may be denoted by $B_n c_m$. As before, if $n > m$, the result is zero; if $n = m$, it is the harmonic invariant; if $n < m$, it is a curve of class $m-n$, which may be called the polar curve of B_n in respect of c_m.

4. Another definition of polar curves may be given. Suppose that the curves g_m, h_n together make up a curve which is harmonic of B_{m+n}. It is convenient to say that the curves g_m, h_n

are *complementary*. Then it is clear that *all the curves complementary to a given curve are harmonic of its polar*. This may be regarded as a generalization of the theorem: All the points conjugate to a fixed point in regard to a conic, lie on the polar of the fixed point.

5. Consider a curve of even order B_{2n}. There is an invariant of the order $\frac{1}{2}(n+1)(n+2)$ in the coefficients, which vanishes when the nth differentials are in involution; this invariant is a symmetrical determinant. Its evectant is a contravariant of the order $2n$ in the variables, and $\frac{1}{2}(n^2+3n)$ in the coefficients, which may be called b_{2n}. The curves B_{2n}, b_{2n} are so related that *if X_n be the polar of y_n in respect of B_{2n}, then the y_n is the polar of X_n in respect of b_{2n}*. That is to say, if $X_n = y_n B_{2n}$, then $X_n b_{2n} = I \,.\, y_n$ where I is the invariant just defined. For example, if we consider a series of conics c_2 and their polars C_2 in regard to a given quartic curve Q_4, then there exists a curve q_4 of the fourth class, such that in respect to it the conics c_2 are the polars of C_2.

6. Two curves whose order and class are different may be made susceptible of the harmonic relation by taking each a proper number of times. Thus, curves B_6, c_3 have an invariant $(B_6)^4 (c_3)^3$, of the order 4 in the coefficients of B_6 and of the order 3 in the coefficients of c_3*. It is to be observed that the equation $B_{mn}(c_m)^n = 0$, is the most general relation of the nth order that can subsist among the coefficients of c_m.

7. The following remarks relate to theorems in Dr Henrici's paper "On certain formulæ concerning the Theory of Discriminants†."

* Thus we may have an invariant which vanishes when a curve is harmonic to itself. Let U_m be the curve, u_n its reciprocal: then $U_m^n u_n^m$ is the invariant in question, or a power of it. For a conic, it is the discriminant; for a cubic, the invariant T of the sixth order. See Salmon's *Higher Algebra*, 1st ed., note to p. 67.

If the curve U_m has no node, the class $= m(m-1)$, and the invariant is $(U_m)^{m-1} u_{m(m-1)}$. If however U_m has a node d, d^2 is part of the reciprocal, and the invariant is $d^2 u_{m(m-1)-2} (U_m)^n$. Cusps may be similarly dealt with.

† [*Proceedings of the London Mathematical Society*, Vol. II. Nos. 15, 16.]

If the polar of C_m in respect of B_n has a node, C_m is harmonic of a curve of order $\zeta m(n-m-1)^2$, which is Dr Henrici's curve $S^{(m)}$. In fact when C_m is a point taken m times, the point is *on* the curve $S^{(m)}$; this is Dr Henrici's theorem.

In general, $x^m y^n B_{m+n+1}$ denotes a straight line. If it vanishes identically, x is a node on the nth polar of y, and y is a node on the mth polar of x. In this case $x^m y^{n-1} B_{m+n-1}$ and $x^{m-1} y^n B_{m+n-1}$ are conics having nodes at y and x respectively*. In the relation

$$x^m y^n B_{m+n+1} \equiv 0$$

write $x+\delta x$ for x and $y+\delta y$ for y. This is equivalent to supposing δx and δy to be points on the tangents at x and y to the curves $S^{(m)}$ $S^{(n)}$ which are the loci of those points respectively. Then we have

$$(my\delta x + nx\delta y)\, x^{m-1} y^{n-1} B_{m+n+1} \equiv 0;$$

operate on this with y; we know that

$$n\delta y \,.\, x^m y^n B_{m+n+1} = 0,$$

it follows that

$$m\delta x \,.\, x^{m-1} y^{n+1} B_{m+n+1} = 0;$$

that is to say, the tangent at x to $S^{(m)}$ is the line-polar of y in respect of the $(m-1)$th polar of x. This is another of Dr Henrici's theorems. I have added this proof as an example of the readiness with which the operative notation lends itself to such investigations.

* Viz., these are the pairs of tangents at the two nodes. It is observable that the tangents at x to the nth polar of y, the tangent to the curve $S^{(m)}$, and the line xy, form a harmonic pencil.

XIV.

ON SYZYGETIC RELATIONS AMONG THE POWERS OF LINEAR QUANTICS*.

In his *Géométrie de Direction* (Paris, 1869), M. Paul Serret makes very beautiful use of a principle which he states nearly as follows (p. 138):

"*In order that a system of points (in a plane) may be so related that every curve of order m passing through all but one of them must pass through the remaining one, it is necessary and sufficient that the* m^{th} *powers of their distances from an arbitrary line should satisfy a linear homogeneous relation*

$$\lambda_1 P_1^m + \lambda_2 P_2^m + \lambda_3 P_3^m + \ldots\ldots \equiv 0\dagger.$$"

There is, of course, an analogous theorem for surfaces, and in fact M. Serret combines the two enunciations into one; he states also the correlative theorems concerning a system of lines or planes such that every curve or surface touching all but one of them, touches also the remaining one. For the sake of clearness I have here stated in full only one of these four theorems.

By the use of Professor Sylvester's method of Contravariant Differentiation I have arrived at certain extensions of these theorems, which I now proceed to explain:—

Theorem I. *In order that a system of N points in a plane should all lie on a curve of order n, it is sufficient that the* p^{th}

* [From the *Proceedings of the London Mathematical Society*, Vol. III. No. 21, pp. 9—12.]

† In the *Bulletin des Sciences Mathématiques et Astronomiques*, January, 1870, M. Darboux observes that this theorem, for the special case $m=2$, had been given by Hesse, *Vier Vorlesungen aus der analytischen Geometrie*, Leipzig, 1866.

powers of their distances from an arbitrary line should satisfy a linear homogeneous relation; the number N being given by the formula

$$N = \tfrac{1}{2}\alpha n(n+3) + \tfrac{1}{2}(\beta+1)(\beta+2),$$

where α is the quotient and β the remainder of the division of p by n, so that $p = \alpha n + \beta$, and $\beta < n$.

Theorem II. *In order that a system of N points in space should all lie on a surface of order n, it is sufficient that the pth powers of their distances from an arbitrary plane should satisfy a linear homogeneous relation; the number N being given by the formula*

$$N = \tfrac{1}{6}\alpha n(n^2+6n+11) + \tfrac{1}{6}(\beta+1)(\beta+2)(\beta+3),$$

where as before

$$p = \alpha n + \beta, \quad \beta < n.$$

To render the nature of these theorems somewhat more clear, I add the following tables of the values of N for given values of p and n:—

TABLE A.—CURVES.

Values of p.	2	3	4	5	6	7	8	9	10	11	12
Line	5	7	9	11	13	15	17	19	21	23	25
Conic.....	6	8	11	13	16	18	21	23	26	28	31
Cubic.....		10	12	15	19	21	24	28	30	33	37
Quartic			15	17	20	24	29	31	34	38	43
Quintic				21	23	26	30	35	41	43	46
Sextic					28	30	33	37	42	48	55
Septic						36	38	41	45	50	56
Octavic							45	47	50	54	59

TABLE B.—SURFACES.

Values of p.	2	3	4	5	6	7	8	9	10	11	12
Plane	7	10	13	16	19	22	25	28	31	34	37
Quadric.....	10	13	19	22	28	31	37	40	46	49	55
Cubic		20	23	29	39	42	48	58	61	67	77
Quartic			35	38	44	54	69	72	78	88	103
Quintic				56	59	65	75	90	111	114	120
Sextic					84	87	93	103	118	139	167
Septic						120	123	129	139	154	175
Octavic							165	168	174	184	199

Here, for example, in the first table opposite the word Cubic and under the power 5 we find the number 15; the theorem corresponding to this is—

If 15 points are such that every quintic through 14 of them passes through the remaining one, all these points must lie on a cubic curve.

Now if we take 15 points arbitrarily on a cubic curve, it is not in general true that the fifth powers of their distances from an arbitrary line satisfy a linear homogeneous relation. That this may be the case, the 15 points must be intersections of the cubic with a quintic; and these are not arbitrary points, but 14 of them being given, the 15th is determined, by a theorem of Jacobi and Plücker. The theorem immediately derived from the table, then, must be completed by this statement; the points are not only all on a cubic, but they are intersections of a cubic and a quintic.

It is to be understood also that if we take a number N of points lying between any two adjacent numbers in the same vertical column of the table, then the same theorem is true about N that is true about the greater of these numbers. Thus we are informed by the first table that a syzygy among the 4th powers of the distances of 12 points makes them lie on a cubic, and that a similar syzygy for 15 points makes them lie on a quartic; this latter theorem is true for the intermediate numbers 13 and 14. It is not however *all* that is true in either of these cases; the 14 points are points of intersection of two quartics, and the 13 points are (I believe) points on a cubic such that no twelve of them are intersections of the cubic with a quartic. I wish particularly to draw attention to these intermediate cases, where it appears that more is true than can be proved by the method to be presently explained.

Method of Demonstration. Let the tangential equation of a point be

$$0 = \alpha\xi + \beta\eta + \gamma\zeta \ (\equiv p, \text{ say})$$

and let the equation of a curve of the nth order be

$$0 = (*\dot{x}, y, z)^n \ (\equiv B_n, \text{ say})$$

then I say that

$$(*\Theta\frac{\delta}{\delta\xi}, \frac{\delta}{\delta\eta}, \frac{\delta}{\delta\zeta})^n . (\alpha\xi + \beta\eta + \gamma\zeta)^n = (*\Theta\alpha, \beta, \gamma)^n \lfloor n ;$$

that is to say, *if we operate with* B_n *on the* n^{th} *power of* p, *we shall obtain the result of substituting the coordinates of* p *for* x, y, z *in* B . If, then, this result vanishes, the point p is on the curve B_n.

I will now prove that if the 12th powers of the *nil-facta* in the tangential equations of 43 points are connected by a linear syzygy, the 43 points are on a quartic curve. We can draw a quartic B_4 through 14 of the points; operate with B_4 on the syzygy, then these 14 points are cleared away, and there remains a syzygy between the 8th powers of the remaining 29 points. We have therefore now to prove that these 29 points are on a quartic. Draw a curve C_4 through 14 of them, and operate on the new syzygy with C_4. This clears away 14 more points, and we are left with a syzygy among the 4th powers of 15 points. But then by Serret's theorem these lie on a quartic. Hence, *any* 15 of the original 43 points are on the same quartic; therefore all the 43 are on the same quartic.

To prove that if the cubes of 13 points in space are connected by a syzygy they lie on a quadric surface, operate with the plane through three of them; we are then left with a syzygy among the squares of 10 points, and Serret's theorem again applies.

The application of this method to the remaining cases will now be easy.

*XV.

ON SYZYGETIC RELATIONS CONNECTING THE POWERS OF LINEAR QUANTICS.

I THINK the first treatment of this subject is to be found in some very interesting articles of M. Paul Serret's; *Nouvelles Annales*, t. IV. (1865), pp. 145, 193, and 433. M. Serret's attention was confined to the *squares* of linear quantics; and in regard to these he establishes such propositions as the following:—If the squares of the characteristics of the equations of six lines satisfy a syzygetic relation, the six lines touch a conic section. That is to say, if $P_1=0$, $P_2=0$, ... $P_6=0$ are the equations of the lines, and if we have an identical relation

$$\lambda_1 P_1^2+\lambda_2 P_2^2+\lambda_3 P_3^2+\lambda_4 P_4^2+\lambda_5 P_5^2+\lambda_6 P_6^2\equiv 0;$$

or, as he finds it convenient to write

$$\Sigma_1^6 \lambda P^2\equiv 0$$

where the λs are numerical coefficients, then the lines P_1 P_2 ... P_6 touch the same conic. Another of his propositions is that if eight planes P_1, P_2 ... P_8 satisfy an identical relation

$$\Sigma_1^8 \lambda P^2\equiv 0,$$

then the eight planes are such that any quadric surface touching seven of them touches also the eighth. These propositions are arrived at by a somewhat circuitous path, though the steps severally are elegant. From the latter M. Serret obtains a very beautiful and immediate proof of Hesse's theorem that two tetrahedra self-conjugate to the same quadric are such that every quadric touching seven of their faces touches also the eighth. Namely, the equation of the first quadric may be written in either of the forms

$$\lambda_1 P_1^2+\lambda_2 P_2^2+\lambda_3 P_3^2+\lambda_4 P_4^2=0,\ \lambda_5 P_5^2+\lambda_6 P_6^2+\lambda_7 P_7^2+\lambda_8 P_8^2=0,$$

and these two being identical to a factor *près*, we have a syzy-

getic relation among the eight squares, from which by the second of the above propositions the theorem in question at once follows.

Having, by an application of Prof. Sylvester's most powerful method of contravariant differentiation, succeeded in extending these propositions to higher powers of linear quantics, and to curves and surfaces of any order, I found as a particular result that two quadrilaterals of the same system totally inscribed in a cubic are such that every curve of the third class touching seven of their sides touches also the eighth. Doubtful of this proposition, I communicated it to the Mathematical Society, and was subsequently informed by Mr Cotterill that the eight lines in question touch the same conic. This is equivalent to the analytic theorem, "if the cubes of eight linear quantics are syzygetic, the squares of any six of them are syzygetic." The proof of this by contravariant differentiation and the statement of a series of analogous propositions occupy the following notes.

I.

If in the tangential equation of a curve

$$c_p \equiv (\xi, \eta, \zeta)^p = 0$$

we write

$$\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz} \text{ for } \xi, \eta, \zeta,$$

and operate upon

$$(lx + my + nz)^p,$$

we shall get $\lfloor p$ multiplied by the result of substituting l, m, n for ξ, η, ζ in c_p; that is to say

$$\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)^p . (lx + my + nz)^p = \underline{p} . (l, m, n)^p.$$

For shortness, denote $(lx + my + nz)$ by Q, and let c_p mean also the differential operator

$$\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)^p .$$

Then *if the operator* c_p *reduces* Q^p *to zero, the line* Q *touches the curve* c_p, *and conversely.*

Suppose now that there are several lines $Q_{(1)}$, $Q_{(2)}$, ... and that there is an identical relation,

$$\Sigma\lambda Q^p \equiv 0,$$

connecting the pth powers of the quantities Q. Let also c_p be a curve touching all but one of the lines, so that the operator c_p reduces to zero all but one of the quantities Q^p. The expression $\Sigma\lambda Q^p$ being identically zero, the result of operating upon it with c_p must be zero, or we have

$$\Sigma\lambda c_p Q^p = 0.$$

But of the terms $\lambda_1 c_p Q_{(1)}^p$, $\lambda_2 c_p Q_{(2)}^p$, ... we know that all vanish but one; it follows that this last one also vanishes, or the curve c_p touches the remaining line. We may therefore enunciate the proposition:—*If there are* n *lines* $Q_1, Q_2, \ldots Q_n = 0$, *and if there is an identical relation*

$$\Sigma_1^n \lambda Q^p \equiv 0,$$

then every curve c_p *of class* p *which touches* $n-1$ *of the lines will also touch the* n^{th}.

It will be sufficient merely to enunciate the obviously corresponding proposition in three dimensions:—

If there are n *planes* $Q_1, Q_2, \ldots Q_n = 0$, *and if there is an identical relation*

$$\Sigma_1^n \lambda Q^p = 0,$$

then every surface c_p *of class* p *which touches* $n-1$ *of the planes will also touch the* n^{th}.

In solid geometry, however, as usual, the analogy branches off into two distinct directions, and we are led to consider a somewhat different theory.

Let the number of straight lines which can be drawn through a fixed point and in a fixed plane to touch a given surface be called the *rank* of the surface (viz. this is both the class of a general plane section and the order of a general tangent cone), then that relation between the six coordinates of a line which expresses that the line touches the surface will be of a degree equal to the rank of the surface. I shall denote the expression equated to zero in this equation by a Greek letter whose suffix

indicates the rank; thus, for example, $\beta_2 = 0$ is the rank-equation* of a quadric surface.

The six coordinates being a, b, c, f, g, h, where $af + bg + ch = 0$, it is very easy to prove that if in β_n an expression of the nth degree in these coordinates, we substitute for a, b, c, f, g, h respectively

$$\frac{d}{df}, \frac{d}{dg}, \frac{d}{dh}, \frac{d}{da}, \frac{d}{db}, \frac{d}{dc},$$

we shall get an invariant symbol of operation. I shall use β_n to mean not only the function of the coordinates, but also this operator obtained by the substitution just defined. This being so, if we call the condition that the line $(abcfgh)$ shall meet a given line σ or $(lmnpqr)$, the *equation* of the line σ (namely the equation is

$$\sigma \equiv pa + qb + rc + lf + mg + nh = 0),$$

then *the condition that* σ *shall touch the surface* β_n *is*

$$\beta_n \sigma^n = 0.$$

From this it follows that

If there are n *straight lines* $\sigma_{(1)}, \sigma_{(2)}, \ldots \sigma_{(n)}$, *and if there is an identical relation*

$$\Sigma_1^n l\sigma^p = 0,$$

then every surface β_p *of rank* p *which touches* $n-1$ *of the lines will also touch the* n^{th}.

II.

At this point I digress somewhat to consider the interpretation of what I have elsewhere called the harmonic invariant of two curves or surfaces, the order of one being equal to the class of the other. First, in the case of two conics, the point-equation of the first being $B_2 = 0$, and the line-equation of the second $c_2 = 0$, the harmonic invariant is $c_2 B_2$, which is commonly called the invariant Θ. Suppose that this vanishes; then if B_2 can be written in the form

$$X^2 + Y^2 + Z^2$$

* [The expression "line-equation" would have been the more natural one, but a confusion might arise between this line-equation of a surface, and the line-equation of a plane curve.—C.]

(so that the lines XYZ form a self-conjugate triangle), since the operator c_2 reduces this to zero we see that if the conic c_2 touch two of the lines it must touch also the third. Similarly, if B_2 can be written in the form

$$X^2 + Y^2 + Z^2 + W^2$$

(so that $XYZW$ form a self-conjugate quadrilateral), if c_2 touches three of the lines it must touch also the fourth. Hence $c_2 B_2 = 0$ is the condition both (1) that c_2 shall be inscribed in an infinite number of *triangles* self-conjugate to B_2, and (2) that c_2 shall be inscribed in an infinite number of *quadrilaterals* self-conjugate to B_2. These are known interpretations; the latter, given first I think by Dr Salmon under a slightly different form (*Conics*, [§ 375]), was reduced to this more simple and natural shape by Professor Cremona (*Educational Times*, Reprint)*. Next, in the case of two quadric surfaces $c_2 = 0$ and $B_2 = 0$, if $c_2 B_2 = 0$, and B_2 can be expressed in either of the forms

$$\Sigma_1^4 . X^2, \ \Sigma_1^5 . X^2, \ \Sigma_1^6 . X^2,$$

we see at once that if c_2 touch all but one of the planes X it must touch also that other. Hence $c_2 B_2$ is the condition that c_2 shall be inscribed (1) in an infinity of *tetrahedra* self-conjugate to B_2, (2) in an infinity of *pentahedra* self-conjugate to B_2, (3) in an infinity of *hexahedra* self-conjugate to B_2. The terms *conjugate hexahedron*, *conjugate pentahedron*, are introduced by M. Serret, and seem likely to be of considerable use.

Passing now to the rank-equations of the two quadrics, which I shall write $\beta_2 = 0$, $\gamma_2 = 0$, I observe that if β_2 can be thrown into the form $\Sigma_1^6 \sigma^2$,
then the 6 lines σ are such that each is conjugate to all the rest; or the lines are the edges of a self-conjugate tetrahedron. If then the harmonic invariant $\gamma_2 \beta_2$ (Dr Salmon's invariant T) vanishes, and γ_2 touches five of these lines, it will touch the sixth; or γ_2 can touch the edges of an infinite number of tetrahedra self-conjugate to β. We have not yet studied the properties relative to β_2 of a system of lines such that β_2 may be expressed in the form $\Sigma_1^p \sigma^2$ when p is greater than 6; yet it is

* [Cf. Vol. IV. p. 109; Vol. IX. pp. 62, 74.]

obvious that such systems will give new interpretations of the invariant T.

The general extension of this method of interpretation is now perfectly easy. *If* B_n *is a curve of* n^{th} *order harmonic of* c_n *a curve of* n^{th} *class, and if* B_n *can be written in the form* $\Sigma_1^p . X^n$, *then if* c_n *touch* $p-1$ *of the lines* X *it will touch also the* p^{th}. Similarly for surfaces in regard to lines and planes. The converse proposition is

The curve or surface $\Sigma_1^p X^n = 0$ *is harmonic of every curve or surface* c_n *of the* n^{th} *class which touches all the lines or planes* X.

I forbear to state the correlative propositions in which lines and planes are replaced by points.

III.

Let us return to the original question.

If there is an identical relation

$$\Sigma_1^8 \lambda P^3 \equiv 0$$

between the cubes of the 8 *linear quantics* P, *there shall also be an identical relation between the squares of any* 6 *of them.*

For we know that we can find a linear differential operator which shall reduce any two of the quantics to zero; namely, let h be the point of intersection of the lines $P_{(7)}$, $P_{(8)}$, having for coordinates a, b, c, then the operator

$$a\frac{d}{dx} + b\frac{d}{dy} + c\frac{d}{dz}$$

being also denoted by h, we have $hP_{(7)} = 0$, $hP_{(8)} = 0$; and so if we operate with h on the given syzygy we obtain therefrom

$$3\Sigma_1^6 \lambda . hP . P^2 \equiv 0,$$

a syzygy connecting the squares of the quantics $P_{(1)}$, $P_{(2)}$ ... $P_{(6)}$.

If there is an identical relation

$$\Sigma_1^{11} \lambda P^4 = 0$$

between the fourth powers of the 11 *linear quantics* P, *there shall also be an identical relation between the squares of any* 6 *of them.*

Let c_2 be the conic section touching the five lines $P_{(7)} \ldots P_{(11)}$; then $c_2 P_{(7)}^2 = 0, \ldots c_2 P_{(11)}^2 = 0$. If then we operate with c_2 on the given syzygy we shall obtain

$$12\Sigma_1^6 \lambda . c^2 P^2 . P^2 \equiv 0,$$

a syzygy connecting the squares of the quantics $P_{(1)}, \ldots P_{(6)}$; provided all the $c_2 P^2$ do not vanish. If however all these vanish, all the lines touch the conic c_2, and there is again a syzygy connecting the squares of any 6 of them. Using these two demonstrations as samples, we are enabled to construct the following table.

Powers $p=$	$n=2$	3	4	5	6	7	8	9	10	11
2	6									
3	8	10								
4	11	12	15							
5	13	15	17	21						
6	16	19	20	23	28					
7	18	21	24	26	30	36				
8	21	24	29	30	33	38	45			
9	23	28	31	35	37	41	47	55		
10	26	30	34	41	42	45	50	57	66	
11	28	33	38	42	48	50	54	60	68	78

*

* [This is in effect the Table A of paper XIV., where it is explained that the number in the Table is

$$N = \frac{\alpha}{2} n(n+3) + \frac{1}{2}(\beta+1)(\beta+2),$$

where α is the quotient and β the remainder of the division of p by n, so that $p = \alpha n + \beta$, and $\beta < n$.

It may be remarked that so long as p is not greater than n, that is down to the bar in each column of the Table, the Number

$$= \tfrac{1}{2} n(n+3) + \tfrac{1}{2}(p-n+2)(p-n+1),$$

and that the several columns are then continued as follows:

Col. $n=2$.	$n=3$.	&c.
6		
8	10	
$11 = 6+5$	12	
$13 = 8+5$	15	
$16 = 11+5$	$19 = 10+9$	
$18 = 13+5$	$21 = 12+9$	
$21 = 16+5$	$24 = 15+9$	
&c.	&c.	

It is easy to see that this is in fact an equivalent construction of the table. C.]

*XVI.

[ON THE THEORY OF DISTANCES.]

[PRELIMINARY*.]

I EXPLAIN in the first place the notation employed, which is an extension of the Geometric Analysis of GRASSMANN, explained by him in the "Ausdehnungslehre" and in Crelle's Journal, and founded in part on a remark of Leibnitz.

GRASSMANN employs single large letters, as A, B, C, to represent straight lines in a plane, and single small letters, as a, b, c, to represent points. When two large letters come together, as AB, the notation is taken to mean the point of intersection of the lines A, B. So when two small letters come together, as ab, the notation is taken to mean the line joining the points a, b. The equation $ABC = 0$ means that the lines A, B, C meet in a point; the equation $abc = 0$, that the points a, b, c lie in a line; and the equation aB or $Ba = 0$, that the point a lies on the line B. No signification is given to the separated symbols ABC, abc, aB, except as equated to zero. The main principle of the application of this method is that the order of an equation in any letter contained is measured by the number of times that letter occurs; a remark which will be further explained in the sequel.

I now explain the extensions of this notation which I have found it convenient to make.

* [The *Preliminary* matter forms the substance of notes given to Prof. Henrici by the Author at the British Association Meeting in the year 1869. The paper itself (pp. 134—157), without a title, appears to have been written subsequently. The title I have given to the two communications has been taken from that of a paper Prof. Clifford read at the above meeting, an abstract of which is given below (p. 164). I have employed ι to represent $\sqrt{-1}$.]

A curve may be given by its points, or by its tangents; that is to say, we may know its equation in point-coordinates (x,y,z), the degree of the equation being the *order* of the curve; or we may know its equation in the contravariant, tangential, or line-coordinates (ξ, η, ζ), the degree of the equation being then the *class* of the curve. This being so, I denote by a large letter a quantic in (x,y,z), and I write the order of the quantic in the form of a suffix. Thus C_3 denotes a cubic in (x,y,z), and $C_3 = 0$ is the equation to a curve of the third order. Next, I use a small letter with a suffix to denote a quantic in (ξ, η, ζ), the order of the quantic being denoted by the suffix. Thus b_3 denotes a cubic in (ξ, η, ζ), and $b_3 = 0$ is the equation to a curve of the third class. When there is no suffix, the suffix 1 is to be understood; in this respect the notation coincides with that of GRASSMANN.

When two large letters come together, each is raised to the power denoted by the suffix of the other, as $A_m^n B_n^m$. The symbol then denotes a quantic which, equated to zero, gives the equation in tangential coordinates of the mn intersections of the curves A_m, B_n. Similarly $a_m^n b_n^m = 0$ is the equation of the mn common tangents of the curves a_m, b_n. The reason of the indices is now apparent; such equation being of the degree n in the coefficients of the first curve, and of the degree m in those of the second.

When three large letters or three small letters come together, each is raised to a power denoted by the product of the suffixes of the other two; the symbol then denotes the resultant of the three quantics.

When a small letter comes before a large one, as $b_m C_n$, the notation is taken to mean the result of changing (ξ, η, ζ) in b_m into $(\partial_x, \partial_y, \partial_z)$ and performing the operation thus indicated on C_n. So, finally, when a large letter comes before a small one, as $B_m c_n$, the notation is taken to mean the result of changing (x, y, z) in B_m into $(\partial_\xi, \partial_\eta, \partial_\zeta)$ and performing the operation thus indicated on c_n.

In the use of these symbols to investigate the relations of geometrical magnitudes, it is to be observed that the absolute

values of (xyz) or $(\xi\eta\zeta)$ or the coefficients of a quantic are not given, but only their ratios; and consequently that the symbols defined above can have no special value but zero. If however we form a fraction such that every letter mentioned occurs an equal number of times in the numerator and denominator, this *will* have a definite numerical value, being a function of the known ratios aforesaid; and may accordingly represent a geometrical magnitude. This theory of *characteristics* is due to Prof. SYLVESTER.

It is further to be observed that the metric properties of figures in plane geometry depend upon the circular points at infinity, which I denote by i, j; and those of figures in spherical geometry upon the imaginary circle at infinity, which I denote by O_2 or o_2 according as it is given by points or tangents. The points i, j in the one case, and the circle O_2 in the other, have received the name of "the Absolute" from Prof. CAYLEY, to whom this theory is due *.

FORMULÆ FOR A PLANE CONIC.

Expressions are obtained below for the *distance* of a point from a conic given tangentially, and for the *distance* of a line from a conic given by its points. Two different geometrical definitions are obtained for each of these; their ratio is a quantity which I have called the *distance of the curve from the absolute.*

The asymptotes of the conic are denoted by P, Q; a pair of foci, viz. either the two real or the two imaginary foci, are denoted by p, q; the conic is called C_2 or c_2.

DISTANCE OF THE POINT a FROM c_2.

Let any straight line B be drawn through the point a, meeting the conic in l, m; let also the tangents from a to the conic be L, M.

* [$\overline{ab}$ = distance between points a and b, ab = line joining a and b.]

First Dist. $a, c_2 = \sin^2 LM \,.\, ap^2 \,.\, aq^2$

$$= \frac{a^2 C_2 \,.\, (a\dot{i}j)^2}{a\dot{i}^2 c_2 \,.\, a\dot{j}^2 c_2} \,.\, \frac{a\dot{i}^2 c_2 \,.\, a\dot{j}^2 c_2}{(aij)^4 \,.\, (\dot{i}\dot{j}^2 c_2)^2} = \frac{a^2 C_2}{(aij)^2 \,.\, (\dot{i}\dot{j}^2 c_2)^2} \,.$$

Second Dist. $a, c_2 = al \,.\, am \,.\, \sin BP \,.\, \sin BQ$

$$= \frac{a^2 C_2 \,.\, (Bi \,.\, Bj)}{(aij)^2 \,.\, (B\dot{i}j)^2 C_2} \,.\, \frac{(B\dot{i}j)^2 C_2}{(Bi \,.\, Bj) \,.\, \sqrt{(i^2 C_2 \,.\, j^2 C_2)}}$$

$$= \frac{a^2 C_2}{(aij)^2 \,.\, \sqrt{(i^2 C_2 \,.\, j^2 C_2)}} \,.$$

The ratio of these two is

$$\frac{\sin^2 LM \,.\, ap^2 \,.\, aq^2}{al \,.\, am \,.\, \sin BP \,.\, \sin BQ} = \frac{\sqrt{(i^2 C_2 \,.\, j^2 C_2)}}{(\dot{i}\dot{j}^2 c_2)^2} = pq^2.$$

Distance of the line A from C_2.

Let any point b be taken on the line A, the tangents from b to the conic being LM; also let A meet the conic in the points l, m.

First Dist. $A, C_2 = lm^2 \,.\, \sin^2 AP \,.\, \sin^2 AQ$

$$= \frac{A^2 c_2 \,.\, Ai \,.\, Aj}{(A\dot{i}j)^2 C_2} \,.\, \frac{(A\dot{i}j)^2 C_2}{(Ai \,.\, Aj)^2 \,.\, i^2 C_2 \,.\, j^2 C_2}$$

$$= \frac{A^2 c_2}{Ai \,.\, Aj \,.\, i^2 C_2 \,.\, j^2 C_2} \,.$$

Second Dist. $A, C_2 = \sin AL \sin AM \,.\, bp \,.\, bq$

$$= \frac{A^2 c_2 \,.\, (b\dot{i}j)^2}{Ai \,.\, Aj \,\sqrt{(b\dot{i}^2 c_2 \,.\, b\dot{j}^2 c_2)}} \,.\, \frac{\sqrt{(b\dot{i}^2 c_2 \,.\, b\dot{j}^2 c_2)}}{(bij)^2 \,.\, \dot{i}\dot{j}^2 c_2}$$

$$= \frac{A^2 c_2}{Ai \,.\, Aj \,.\, \dot{i}\dot{j}^2 c_2} \,.$$

The ratio of these two is

$$\frac{lm^2 \,.\, \sin^2 AP \,.\, \sin^2 AQ}{\sin AL \sin AM \,.\, bp \,.\, bq} = \frac{\dot{i}\dot{j}^2 c_2}{i^2 C_2 \,.\, j^2 C_2} = \sin^2 PQ.$$

In the case of a sphero-conic we obtain analogous expressions for the distance of a point or line (great circle) from the conic, but the value depends on the pair of foci or cyclic planes selected; the ratio of such different values is however the same for all points and lines. Moreover, the ratio of the two distances of a point or of a line is a quantity independent of the point or line, but I have as yet obtained no geometrical definition of it. For this reason I have not treated separately the formulæ for a sphero-conic, which are of course like the preceding included in the general formulæ of curves*.

I.

All magnitudes which are concerned in plane geometry may be expressed in terms of three, which on this account are of primary importance. These are the distance of two points, the distance of a point from a line, and the sine of the angle between two lines. We obtain the most simple expressions for these magnitudes by employing rectangular Cartesian co-ordinates for the points, and the coordinates of Dr Booth for the lines; but it is convenient from the first to make these expressions homogeneous by the introduction of a third co-ordinate which may be put $=1$ or -1 in the two cases respectively. Thus if a_1, a_2, a_3 are the coordinates of the point a, $\frac{a_2}{a_3}$ and $\frac{a_1}{a_3}$ are its distances from the axes; and if A_1, A_2, A_3 are the coordinates of a line A, $-\frac{A_3}{A_1}$ and $-\frac{A_3}{A_2}$ are the intercepts it cuts off from the axes. This being so, the expressions for our primary magnitudes are

$$\text{Dist. } ab = \frac{\sqrt{\{(a_1b_3 - a_3b_1)^2 + (a_2b_3 - a_3b_2)^2\}}}{a_3b_3},$$

$$\text{Dist. } aB = \frac{a_1B_1 + a_2B_2 + a_3B_3}{a_3\sqrt{(B_1^2 + B_2^2)}},$$

$$\sin AB = \frac{A_1B_2 - A_2B_1}{\sqrt{(A_1^2 + A_2^2)}\sqrt{(B_1^2 + B_2^2)}},$$

* [Cf. however V., p. 152 *infra*, of this paper.]

and these are clearly not invariant in regard to the points and lines. Let us now ask what is the locus of the points b which are at zero distance from a. We find that it consists of the two straight lines

$$a_3b_1 + \iota\, a_3b_2 - (a_1 + \iota\, a_2)\, b_3 = 0,$$
$$a_3b_1 - \iota\, a_3b_2 - (a_1 - \iota\, a_2)\, b_3 = 0.$$

Each of these lines passes through the point a; thus we learn that *the lines of null-length are straight lines, and two of them pass through every point of the plane.*

If the point a moves about, each of the lines of null-length remains parallel to the same direction, or, which is the same thing, passes through a fixed point at an infinite distance. Let these two points be called i, j; their coordinates may be taken to be

$$i_1 : i_2 : i_3 = \frac{1}{\sqrt{2}} : \frac{\iota}{\sqrt{2}} : 0,$$

$$j_1 : j_2 : j_3 = \frac{\iota}{\sqrt{2}} : \frac{1}{\sqrt{2}} : 0,$$

so that the line ij has coordinates (0, 0, 1). Then we have

$$abi = \frac{1}{\sqrt{2}} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 1 & \iota & 0 \end{vmatrix} = \frac{1}{\sqrt{2}} \{(a_2b_3 - a_3b_2) - \iota\, (a_1b_3 - a_3b_1)\},$$

$$abj = \frac{1}{\sqrt{2}} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \iota & 1 & 0 \end{vmatrix} = \frac{1}{\sqrt{2}} \{\iota\, (a_2b_3 - a_3b_2) - (a_1b_3 - a_3b_1)\},$$

$$aij = a_3, \quad bij = b_3,$$

$$abi \,.\, abj = \frac{\iota}{2} \cdot \{(a_2b_3 - a_3b_2)^2 + (a_1b_3 - a_3b_1)^2\}$$

and therefore

$$\kappa \,.\, \text{Dist. } ab = \frac{\sqrt{(abi \,.\, abj)}}{aij \,.\, bij}, \quad \text{where } \kappa^2 = \frac{\iota}{2}.$$

Thus we learn that *all the lines of null-length pass through one or other of the points i, j;* and *the distance ab may be expressed in terms of the invariants of a, b, i, j* to a numerical

factor *près.* This factor κ is the same for all distances, depending only upon the absolute value given to the coordinates of i, j. To reduce κ to the value unity, we have only to multiply these coordinates throughout by $\kappa^{-\frac{1}{2}}$.

Similar expressions may now be found for the other two primary magnitudes. We have in fact

$$Ai = \frac{1}{\sqrt{2}}(A_1 + \iota A_2), \quad Aj = \frac{1}{\sqrt{2}}(\iota . A_1 + A_2),$$

$$Ai . Aj = \frac{\iota}{2} . (A_1^2 + A_2^2),$$

and thence

$$\left[\frac{1}{\kappa}\right] \text{Dist. } aB = \frac{aB}{aij . \sqrt{(Bi . Bj)}}$$

$$-2\iota . \sin AB = \frac{AB\bar{ij}}{\sqrt{(Ai . Aj . Bi . Bj)}}.$$

(The last expression may be simplified as follows:

We have

$$AB\bar{ij} = Ai . Bj - Aj . Bi$$

by a well-known theorem of determinants; but also

$$Ai . Bj + Aj . Bi$$

$$= \tfrac{1}{2}\{(A_1 + \iota A_2)(\iota B_1 + B_2) + (\iota A_1 + A_2)(B_1 + \iota B_2)\}$$

$$= \iota (A_1 B_1 + A_2 B_2)$$

and therefore

$$2 \cos AB = 2 \frac{A_1 B_1 + A_2 B_2}{\sqrt{(A_1^2 + A_2^2)}\sqrt{(B_1^2 + B_2^2)}} = \frac{Ai . Bj + Aj . Bi}{\sqrt{(Ai . Aj . Bi . Bj)}}.$$

If therefore we write θ for the angle AB, we have

$$\cos\theta + \iota \sin\theta = \epsilon^{\theta\iota} = \frac{Aj . Bi}{\sqrt{(Ai . Aj . Bi . Bj)}}$$

$$\cos\theta - \iota \sin\theta = \epsilon^{-\theta\iota} = \frac{Ai . Bj}{\sqrt{(Ai . Aj . Bi . Bj)}},$$

and thence

$$\epsilon^{2\theta\iota} = \frac{Aj . Bi}{Ai . Bj}.$$

This last is an anharmonic ratio in which the lines AB divide the segment ij, and so is an absolute invariant.)

By analogy and for convenience of expression we shall call the sine of the angle between two lines the *distance* of the lines; we may then derive from these formulæ the following theorems:—

If a line pass through either of the points i, j, it is at an infinite distance from all other lines and points, and the distance between any two points on it is zero.

If a point lie upon the line ij, it is at an infinite distance from all other lines and points, and the distance between any two lines through it is zero.

II.

A conic may be given as a curve of the second order C_2 or as a curve of the second class c_2; but either of these expressions may be derived from the other by means of the formula

$$\begin{vmatrix} axC_2, & bxC_2 \\ ayC_2, & byC_2 \end{vmatrix} = 2\overline{ab}\ \overline{xy}\ c_2$$

and its reciprocal

$$\begin{vmatrix} AXc_2, & BXc_2 \\ AYc_2, & BYc_2 \end{vmatrix} = 2\overline{AB}\ \overline{XY}\ C_2.$$

Namely, if we make a, b identical with x, y, and A, B with X, Y, we have

$$2\overline{xy}^2 c_2 = x^2C_2 \,.\, y^2C_2 - (xyC_2)^2$$

and

$$2\overline{XY}^2 C_2 = X^2c_2 \,.\, Y^2c_2 - (XYc_2)^2.$$

The discriminant C_2c_2 may be given in a similar manner. Namely, we have

$$3\overline{xC_2}\ \overline{yC_2}\ \overline{zC_2} = 8xyz \,.\, C_2c_2.$$

The relations of a conic to the points ij are most conveniently expressed in terms of the asymptotes and the foci. The asymptotes P, Q are tangents at the points where the conic

is met by the line ij. The equation to the pair of tangents at the points where a line X meets the conic C_2 must be of the form

$$C_2 - \lambda X^2 = 0.$$

Equating to zero the discriminant of this, we find that

$$C_2 c_2 - 3\lambda X^2 c_2 = 0,$$

whence the equation to the tangents is

$$3C_2 . X^2 c_2 - X^2 . C_2 c_2 = 0.$$

Substituting herein for X the line ij, we obtain for the asymptotes the expression

$$P . Q = 3C_2 . \overline{ij}^2 c_2 - \overline{ij}^2 . C_2 c_2.$$

Thus the product $P . Q$ is of the third order in the coefficients of the conic, whether it is given by its point- or by its line-equation.

To determine the angle between the asymptotes, we observe that the point PQ is the centre of the conic, which is the pole of the line ij, and is therefore represented by $\overline{ij} c_2$. Consequently $PQij$ must be proportional to some power of $\overline{ij}^2 c_2$. If we regard C_2 as initially given, we shall in fact have

$$(PQij)^2 = \kappa (\overline{ij}^2 c_2)^3,$$

each side being of the sixth order in the coefficients. If however c_2 is initially given, the formula becomes

$$(PQij)^2 = \kappa' (\overline{ij}^2 c_2)^3 . C_2 c_2.$$

Next we find by direct substitution

$$Pi . Qi = 3i^2 C_2 . \overline{ij}^2 c_2,$$

$$Pj . Qj = 3j^2 C_2 . \overline{ij}^2 c_2,$$

and therefore

$$\sin^2 PQ = \frac{\overline{ij}^2 c_2}{i^2 C_2 . j^2 C_2} \text{ to a factor } près.$$

But we have also

$$\overline{ij}^2 c_2 = i^2 C_2 . j^2 C_2 - (ijC_2)^2;$$

now it is clear that ijC_2 vanishes when the points ij are conjugate in regard to the conic, or when the asymptotes are at right angles, that is, when $\sin PQ = 1$. It follows therefore that the factor in the last equation is unity, and that

$$\cos PQ = \frac{ijC_2}{\sqrt{(i^2C_2 . j^2C_2)}}.$$

We find in this way that the κ of our formula is $= -36$, so that we may write

$$(PQij)^2 = -36\,(\bar{ij}^2 c_2)^3.$$

This constant might also have been determined by means of the particular case in which the conic breaks up into two straight lines, in which case these lines are themselves the asymptotes. If $C_2 = X.Y$, $-4c_2 = \overline{XY}^2$, and $-4\bar{ij}^2 c_2 = \overline{XYij}^2$; by these formulæ the values just obtained may be compared with the known values of $\sin XY$, $\cos XY$.

If the conic c_2 touches the four lines joining p, q to i, j, we must have

$$\kappa p.q = c_2 + \lambda . i.j.$$

But the equation

$$0 = \text{Disct.}\,(c_2 + \lambda . i.j) = C_2c_2 + \tfrac{3\lambda}{2}\,ijC_2 - \tfrac{3}{4}\lambda^2 \bar{ij}^2 c_2$$

gives two values for λ, one belonging to $p.q$ and the other to $p'.q'$. Hence, eliminating λ, we may write

$$p.q.p'.q' = C_2c_2 . i^2.j^2 - \tfrac{3}{2}ijC_2 . i.j.c_2 - \tfrac{3}{4}\bar{ij}^2 c_2 . c_2^2,$$

a result of the third order in the coefficients of c_2.

If the conic touch the line ij, that is, if $\bar{ij}^2 c_2 = 0$, one of the foci, say q, coincides with the point of contact, and two others, p' and q', coincide with the points i and j respectively; we have therefore in this case $qij = 0$, $p'q'i = 0$, $p'q'j = 0$. If the conic pass through the point i, which happens when $i^2C_2 = 0$, the foci coincide two and two, p with q', suppose, and q with p'; we have then $pqi = 0$, $p'q'i = 0$. Hence the product

$$pqi . pqj . p'q'i . p'q'j$$

will vanish in three cases; (1) when $i^2C_2 = 0$, (2) when $j^2C_2 = 0$,

(3) when $\bar{ij}^2 c_2 = 0$; and it is easy to see that it cannot vanish in any other case. Consequently we must have

$$pqi \cdot pqj \cdot p'q'i \cdot p'q'j = \kappa\,(i^2 C_2 \cdot j^2 C_2)^x\,(\bar{ij}^2 c_2)^y,$$

where a comparison of dimensions gives us the equations

$$4x + y = 6,\ 2x + 2y = 6,\ x = 1,\ y = 2.$$

We have also by direct substitution

$$pij \cdot qij \cdot p'ij \cdot q'ij = \bar{ij}^4\,(p \cdot q \cdot p' \cdot q') = -\tfrac{3}{4}\,(\bar{ij}^2 c_2)^3,$$

and hence the following expression for the product of the squares of the distances pq, $p'q'$,

$$\overline{pq}^2 \cdot \overline{p'q'}^2 = \kappa' \frac{(i^2 C_2 \cdot j^2 C_2)^x\,(\bar{ij}^2 c_2)^y}{(\bar{ij}^2 c_2)^6}.$$

Since $\overline{pq}^2 + \overline{p'q'}^2 = 0$, this product is the same thing as $-pq^4$. But we get another equation between x and y by supposing the conic c_2 to break up into two points u, v, which are then themselves the foci. In that case $-4C_2 = \overline{uv}^2$, $\bar{ij}^2 c_2 = uij \cdot vij$, and the expression for the fourth power of the distance is

$$\overline{uv}^4 = \kappa^4 \frac{(uvi \cdot uvj)^2}{(uij \cdot vij)^4} = \frac{i^2 C_2 \cdot j^2 C_2}{(\bar{ij}^2 c_2)^4},$$

whence $x = 1$, $y = 2$, and the general formula is

$$\overline{pq}^2 = \frac{\sqrt{(i^2 C_2 \cdot j^2 C_2)}}{(\bar{ij}^2 c_2)^2}.$$

In this it is clear that c_2 is primarily given; if C_2 is given, the process of calculating it back from the coefficients of c_2 introduces the factor $C_2 c_2$, and we have

$$\overline{pq}^2 = \frac{\sqrt{(i^2 C_2 \cdot j^2 C_2)}\ C_2 c_2}{(\bar{ij}^2 c_2)^2}.$$

With the aid of this formula and the angle between the asymptotes we may now determine the axes of the conic.

If h and k be the axes, we have

$$\overline{pq}^2 = h^2 - k^2,$$

$$\cos(PQ) = \frac{h^2 + k^2}{h^2 - k^2},$$

and therefore

$$h^2+k^2=\overline{pq}^2\cos PQ=\frac{\sqrt{(i^2C_2.j^2C_2)}.C_2c_2}{(ij^2c_2)^2}.\frac{ijC_2}{\sqrt{(i^2C_2.j^2C_2)}}$$
$$=\frac{ijC_2.C_2c_2}{(ij^2c_2)^2}.$$

Again we have

$$\sin^2PQ=1-\left(\frac{h^2+k^2}{h^2-k^2}\right)^2=-\frac{4h^2k^2}{(h^2-k^2)^2},$$

and therefore

$$-4h^2k^2=\overline{pq}^4\sin^2PQ$$
$$=\frac{i^2C_2.j^2C_2.(C_2c_2)^2}{(ij^2c_2)^4}.\frac{ij^2c_2}{i^2C_2.j^2C_2}$$
$$=\frac{(C_2c_2)^2}{(ij^2c_2)^3}.$$

These last formulæ are of course the well-known ones.

III.

We go on to consider the relations between a point and a conic; and in particular to determine the angle between the tangents from the point to the conic (fig. 12).

The tangents LM from x to the conic C_2 are given by the equation

$$LM=x^2C_2.C_2-(xC_2)^2,$$

we have also

$$-4\sin^2LM=\frac{(LMij)^2}{Li.Lj.Mi.Mj}.$$

The numerator of this fraction vanishes when the intersection of L and M is on the line ij. If these tangents are distinct, the intersection is x; if they coincide, that is, if x is on the conic, or if the conic breaks up, the intersection is indeterminate. Hence $(LMij)^2$ must vanish whenever xij or x^2C_2, or C_2c_2, vanishes; and in no other case. But $(LMij)^2$ is of the fourth order in the coefficients of x and of the conic; therefore

$$(LMij)^2=\kappa x^2C_2.(xij)^2.C_2c_2.$$

The denominator may be expressed by direct operation on LM; we should find in fact

$$Li \,.\, Lj \,.\, Mi \,.\, Mj = \{x^2 C_2 \,.\, i^2 C_2 - (xi C_2)^2\}\,\{x^2 C_2 \,.\, j^2 C_2 - (xj C_2)^2\}$$
$$= \overline{xi^2 c_2} \,.\, \overline{xj^2 c_2},$$

and thus finally

$$\sin^2 LM = \kappa' \frac{x^2 C_2 \,.\, (xij)^2 \,.\, C_2 c_2}{\overline{xi^2 c_2} \,.\, \overline{xj^2 c_2}}.$$

But the denominator may be put into another form which is more useful. Let us assume that the absolute values of the coefficients of p, q, p', q' are so chosen that $pi = p'i$, $qi = q'i$, $pj = q'j$, $qj = p'j$. We have by direct substitution

$$pix \,.\, qix \,.\, p'ix \,.\, q'ix = -\tfrac{3}{4}\overline{ij^2 c_2} \,.\, (\overline{xi^2 c_2})^2,$$

or

$$(pix \,.\, qix)^2 = -\tfrac{3}{4}\overline{ij^2 c_2}\,(\overline{xi^2 c_2})^2.$$

Hence we have

$$xpi \,.\, xqi \,.\, xpj \,.\, xqj = -\tfrac{3}{4}\overline{ij^2 c_2} \,.\, \overline{xi^2 c_2} \,.\, \overline{xj^2 c_2},$$

and consequently, since

$$\overline{xp^2} \,.\, \overline{xq^2} = \kappa \frac{xpi \,.\, xqi \,.\, xpj \,.\, xqj}{(xij)^4\,(pij \,.\, qij)^2},$$

and

$$(pij \,.\, qij)^2 = -\tfrac{3}{4}(\overline{ij^2 c_2})^3,$$

it follows that

$$xp^2 \,.\, xq^2 = \kappa \frac{\overline{xi^2 c_2} \,.\, \overline{xj^2 c_2}}{(xij)^4\,(\overline{ij^2 c_2})^2} = xp'^2 \,.\, xq'^2.$$

Combining these two results, we find

$$xp^2 \,.\, xq^2 \,.\, \sin^2 LM = \kappa'' \frac{x^2 C_2 \,.\, C_2 c_2}{(xij)^2 \,.\, (\overline{ij^2 c_2})^2},$$

a result which may be further simplified by help of the formula for the distance between the foci. Namely, we have

$$\frac{xp^2 \,.\, xq^2 \,.\, \sin^2 LM}{pq^2} = \frac{x^2 C_2}{(xij)^2 \sqrt{(i^2 C_2 \,.\, j^2 C_2)}} \text{ to a factor } \textit{près}.$$

We shall call this quantity the *distance** of the point from the conic; it vanishes when the point is on the conic, and is infinite if either the point or the conic has contact with the absolute.

* [Second distance; cf. p. 133.]

It is to be noted that the conic is given as a curve of the second order, in the form C_2; if it were given in the form c_2, the formula preceding would become

$$xp^2 \,.\, xq^2 \,.\, \sin^2 LM = \kappa'' \frac{x^2 C_2}{(xij)^2 \,.\, (\overline{ij}^2 c_2)^2},^*$$

and this quantity might be taken as the distance of the point from a conic given as a curve of the second class. If the conic breaks up into two lines, the former expression becomes the product of the perpendicular distances of the point from the two lines; if the conic breaks up into two points, the latter expression becomes four times the squared area which they include with the given point. The former expression, however, in which the conic is given as of the second order, admits of a further interpretation, to which we now proceed.

Through the point x (fig. 13) let a line X be drawn, meeting the conic in l, m. For the product of the segments xl, xm we have the formula

$$xl \,.\, xm = \kappa^2 \frac{\sqrt{(xli \,.\, xlj \,.\, xmi \,.\, xmj)}}{(xij)^2 \,.\, lij \,.\, mij}.$$

The numerator of this expression clearly vanishes if x is on the conic, when one of the lines xl, xm becomes indeterminate; otherwise each of these lines is simply the line X. We must have therefore

$$xli \,.\, xlj \,.\, xmi \,.\, xmj = (x^2 C_2 \,.\, Xi \,.\, Xj)^2 \text{ to a factor } \textit{près}.$$

For the denominator we observe that if U is any arbitrary line,

$$lU \,.\, mU = \overline{XU}^2 C_2,$$

and taking the co-ordinates of U for the current line co-ordinates, this gives the tangential equation to l, m. From this we get

$$lij \,.\, mij = \overline{Xij}^2 C_2,$$

and our expression is therefore transformed into

$$xl \,.\, xm = \frac{x^2 C_2 \,.\, Xi \,.\, Xj}{(xij)^2 \,.\, \overline{Xij}^2 C_2}.$$

* [First distance; cf. p. 133.]

The denominator of this will clearly vanish if X is parallel to one or other of the asymptotes P, Q, or if $XPij \,.\, XQij = 0$. Let us, therefore, now seek the product of the distances of X from the asymptotes.

We have

$$-4 \sin XP \,.\, \sin XQ = \frac{XPij \,.\, XQij}{(Xi \,.\, Xj)\sqrt{(Pi \,.\, Pj \,.\, Qi \,.\, Qj)}}.$$

But we find by operating with $(Xij)^2$ upon $P \,.\, Q$ that

$$XPij \,.\, XQij = 3\overline{Xij}^2 C_2 \,.\, \overline{ij}^2 c_2,$$

and moreover

$$Pi \,.\, Pj \,.\, Qi \,.\, Qj = 9i^2 C_2 \,.\, j^2 C_2 \,.\, (\overline{ij}^2 c_2)^2.$$

Hence we have

$$-4 \sin XP \,.\, \sin XQ = \frac{\overline{Xij}^2 C_2}{3Xi \,.\, Xj\sqrt{(i^2 C_2 \,.\, j^2 C_2)}}.$$

Multiplying this by $xl \,.\, xm$, the line X disappears from the result and we find

$$xl \,.\, xm \,.\, \sin XP \,.\, \sin XQ = \frac{x^2 C_2}{(xij)^2\sqrt{(i^2 C_2 \,.\, j^2 C_2)}} \text{ to a factor.}$$

But the expression on the right is the same that we previously obtained for the distance of the point from the conic. Hence the quantity $xl \,.\, xm \,.\, \sin XP \sin XQ$ must be proportional to the quantity $\frac{xp^2 \,.\, xq^2}{pq^2} \sin^2 LM$. To determine the constant factor, suppose the conic to break up into a pair of points; these may be taken to be the points p, q, and the asymptotes will both coincide with the line pq (fig. 14). Here it is clear that $xl \,.\, xm \,.\, \sin XP \sin XQ = xl^2 \,.\, \sin^2 XP =$ squared distance of x from line pq; while $\frac{xp \,.\, xq \sin pxq}{pq} = \frac{\text{twice area } pxq}{pq} =$ distance of x from pq. Thus the factor is unity, and we have always

$$xl \,.\, xm \,.\, \sin XP \,.\, \sin XQ = \frac{xp^2 \,.\, xq^2}{pq^2} \sin^2 LM.*$$

There is no difficulty in investigating the correlative formulæ. First to find the length of the chord cut off a line X

* [Second distance, p. 133.]

by the conic; let lm be the points of intersection, then we have

$$lm^2 = \left[\frac{1}{\kappa^2}\right] \frac{lmi \,.\, lmj}{(lij)^2 \,.\, (mij)^2}.$$

The numerator vanishes if the line X pass through either of the points ij, or if l, m coincide, that is to say, if X touch the conic c_2, or if the conic break up into a pair of points. Moreover, since we have

$$l \,.\, m = X^2 c_2 \,.\, c_2 - (Xc_2)^2,$$

the expression $(lmi)^2$ must be of the fourth order in the coefficients of X and of the conic, and therefore

$$(lmi)^2 = (Xi)^2 \,.\, X^2 c_2 \,.\, C_2 c_2 \text{ to a factor.}$$

Again, we have by direct substitution

$$lij \,.\, mij = X^2 c_2 \,.\, \bar{ij}^2 c_2 - (X\bar{ij}c_2)^2 = \overline{Xij}^2 C_2,$$

and thence

$$lm^2 = \frac{Xi \,.\, Xj \,.\, X^2 c_2 \,.\, C_2 c_2}{(\overline{Xij}^2 C_2)^2} \text{ to a factor.}$$

But we have already found that

$$\sin XP \,.\, \sin XQ = \frac{\overline{Xij}^2 C_2}{Xi \,.\, Xj \sqrt{(i^2 C_2 \,.\, j^2 C_2)}},$$

consequently

[i] $$lm^2 \,.\, \sin^2 XP \,.\, \sin^2 XQ = \frac{X^2 c_2 \,.\, C_2 c_2}{Xi \,.\, Xj \,.\, i^2 C_2 \,.\, j^2 C_2}.$$ *

Lastly, c_2 being primarily given, we have

$$\sin^2 PQ = \frac{\bar{ij}^2 c_2 \,.\, C_2 c_2}{i^2 C_2 \,.\, j^2 C_2},$$

and so

[ii] $$lm^2 \frac{\sin^2 XP \,.\, \sin^2 XQ}{\sin^2 PQ} = \frac{X^2 c_2}{Xi \,.\, Xj \,.\, \bar{ij}^2 c_2}.$$

Let now x be a variable point on the line X (fig. 15), and draw the tangents L, M from x to the conic. Then

$$\sin XL \,.\, \sin XM = \frac{XLij \,.\, XMij}{Xi \,.\, Xj \sqrt{(Li \,.\, Lj \,.\, Mi \,.\, Mj)}}.$$

* [First distance of line X from Conic C_2; cf. p. 133.]

In this fraction the numerator vanishes when X touches the conic, and when x is on the line ij. It must be of the first order in the coefficients of c_2 and of the second in those of x. Hence we have, to a factor *près*,

$$XLij \,.\, XMij = X^2 c_2 \,.\, (xij)^2.$$

Moreover by a previous formula

$$Li \,.\, Lj \,.\, Mi \,.\, Mj = \overline{xi}^2 c_2 \,.\, \overline{xj}^2 c_2.$$

Thus

$$\sin XL \,.\, \sin XM = \frac{X^2 c_2 \,.\, (xij)^2}{Xi \,.\, Xj \,\sqrt{(\overline{xi}^2 c_2 \,.\, \overline{xj}^2 c_2)}}.$$

But, also by a previous formula,

$$xp \,.\, xq = \frac{\sqrt{(\overline{xi}^2 c_2 \,.\, \overline{xj}^2 c_2)}}{(xij)^2 \,.\, \overline{ij}^2 c_2};$$

therefore

$$\text{[iii]} \quad xp \,.\, xq \sin XL \sin XM = \frac{X^2 c_2}{Xi \,.\, Xj \,.\, \overline{ij}^2 c_2}$$

$$= lm^2 \frac{\sin^2 XP \,.\, \sin^2 XQ}{\sin^2 PQ}.*$$

To verify this, suppose the conic to break up into two lines P, Q (fig. 16), which are themselves the asymptotes; the two foci will then coincide at the point PQ, and the tangents LM will pass through the same point. Then $xp \,.\, xq \,.\, \sin XL \,.\, \sin XM = xp^2 \sin^2 XL$ = squared perpendicular from PQ on X. And

$$\frac{lm}{\sin PQ} \,.\, \sin XP \sin XQ = lp \,.\, mp \,.\, \frac{lm}{\sin PQ} \,.\, \frac{\sin XP}{lp} \,.\, \frac{\sin XQ}{mp}$$

$$= \frac{lp \,.\, mp \,.\, \sin PQ}{lm} = \frac{2 \text{ area } lmp}{lm}$$

$$= \text{perpendicular from } PQ \text{ on } X.$$

This quantity†, of which three expressions are given in our last equation, may be called the *distance* of the line X from the conic c_2.

* [Second distance of line X from Conic c_2.]

† [i.e. iii. supra.]

IV.

We now consider a curve C_n of the nth order, which may also be given as a curve $c_{n(n-1)}$ of class $n(n-1)$. To this number $n(n-1)$ there are no reductions in virtue of any singularities that C_n may have; its nodes will enter as double factors and its cusps as triple factors in $c_{n(n-1)}$. This being so, we may write

$$\text{Disct.}_\lambda (x+\lambda y)^n C_n = \frac{\{\underline{|n}\}^{2(n-1)}}{\underline{|n(n-1)}} \cdot \overline{xy}^{n(n-1)} c_{n(n-1)}.$$

Conversely, if we are given a curve c_m of class m, this is also a curve $C_{m(m-1)}$ of order $m(m-1)$; but each double tangent is now a double factor and each stationary tangent a triple factor in $C_{m(m-1)}$. We may gain shortness without introducing confusion, if when C_n is primarily given, we denote $n(n-1)$ by m; and if when c_m is primarily given, we denote $m(m-1)$ by n. Thus in the latter case we shall have

$$\text{Disct.}_\lambda (X+\lambda Y)^m c_m = \frac{\{\underline{|m}\}^{2(m-1)}}{\underline{|n}} \cdot \overline{XY}^n C_n.$$

The curve c_m has m^2 foci, which are the intersections of the m tangents from the point i with the m tangents from the point j. But for every tangent from i there is *one* tangent from j which meets it in a real point; thus there are m real foci. The foci may be arranged in various ways into m sets, such that no two points of the same set are collinear with i or j. By joining the points of any one set with i and j we obtain all the tangents. The real foci constitute a set; and from them we may pass to any other set by successively substituting for each pair pq their *antipoints* $p'q'$, that is to say, the remaining intersections of pi, pj with qi, qj. Now for every such substitution the following equations hold good:

$$p'q'^2 = -pq^2,$$
$$xp' \,.\, xq' = xp \,.\, xq,$$

where x is any point in the plane. Hence if we form the product Πpq^2 of the squared distances of the real foci from one

another, this product can differ only in sign from a similar product formed with any other set; and the same is true of Πxp. The number of possible sets is clearly the same as the number of terms in a determinant of the mth order, viz.: $\underline{|m}$. If we then raise Πpq^2 to the power $\underline{|m}$, we must have some power of $\Pi\Pi pq^2$, the product of the squared distances of all pairs of foci from one another, excepting of course those pairs which are collinear with i or j. But there are $m^2(m-1)^2$ such pairs; thus if k is the power in question

$$\underline{|m} \cdot m(m-1) = km^2(m-1)^2,$$

or
$$k = \frac{\underline{|m}}{m(m-1)},$$

whence
$$(\Pi pq^2)^{m(m-1)} = \Pi\Pi pq^2.$$

In a similar way we find

$$(\Pi xp)^m = \Pi\Pi xp.$$

Now

$$\Pi\Pi pq^2 = \Pi\Pi \frac{pqi \cdot pqj}{pij^2 \cdot qij^2} = \frac{\Pi\Pi pqi \cdot pqj}{(\Pi pij)^{4m^2(m-1)^2}} \text{ [to factor } près\text{]}.$$

To obtain the equation of the foci we may proceed as follows. The equation of the tangents from i, j to c_m is $I_m = 0$, $J_m = 0$, where

$$x^m I_m = \overline{ij}^m c_m, \quad x^m J_m = \overline{jx}^m c_m,$$

and if we then form the equation of the points of intersection of I_m, J_m (which may be written $(XIJ)_{mm} = 0$) it is of the order m in each of them and must contain the equation of the foci. But also it must contain the factor $\overline{ij}^m c_m$, and this to the degree $\overline{m-1}$; for the factor is involved as the condition that I_m should pass through a double point of J_m. Thus the equation of the foci is of the order $2m - \overline{m-1} = m+1$ in c_m and $m^2 - m(m-1) = m$ in i and j. In fact one term in it is a numerical multiple of $\overline{ij}^m c_m \cdot (c_m)^m$.

As in the case of the conic, the product $\Pi\Pi pqi \cdot pqj$ will vanish when $i^n C_n = 0$, or when $j^n C_n = 0$, or when $\overline{ij}^m c_m = 0$. Thus

$$\Pi\Pi pqi \cdot pqj = \kappa (i^n C_n \cdot j^n C_n)^x (\overline{ij}^m c_m)^y.$$

Now the left-hand side is of order $2(m-1)^2$ in the foci and besides of order $\frac{1}{2}m^2(m-1)^2$ in i and j; that is, of order $2(m+1)(m-1)^2$ in c_m, and $2m(m-1)^2+\frac{1}{2}m^2(m-1)^2=\frac{1}{2}m(m+4)(m-1)^2$ in i and j. First regard c_m as given; then C_n is of order $2(m-1)$ in the coefficient of c_m, and we have

$$2(m+1)(m-1)^2 = 4x(m-1) + y,$$

$$\tfrac{1}{2}m(m+4)(m-1)^2 = xm(m-1) + my,$$

whence $x = \frac{1}{2}m(m-1)$, $y = 2(m-1)^2$, and consequently

$$\Pi\Pi pqi \,.\, pqj = \kappa\,(i^n C_n \,.\, j^n C_n)^{\frac{1}{2}m(m-1)}\,(\bar{ij}^m c_m)^{2(m-1)^2}.$$

Next we have

$$\Pi\Pi pij = (\bar{ij}^m c_m)^{m+1} \text{ to a factor,}$$

and therefore

$$\Pi\Pi pq^2 = \frac{\Pi\Pi pqi \,.\, pqj}{(\Pi\Pi pij)^{2(m-1)^2}} = \kappa\,\frac{(i^n C_n \,.\, j^n C_n)^{\frac{1}{2}m(m-1)}}{(ij^m c_m)^{2m(m-1)^2}},$$

but

$$(\Pi pq^2)^{m(m-1)} = \Pi\Pi pq^2;$$

therefore

$$\Pi pq^2 = \frac{(i^n C_n \,.\, j^n C_n)^{\frac{1}{2}}}{(ij^m c_m)^{2(m-1)}}.$$

We now proceed to determine Πxp, where x is any point in the plane. We have

$$\Pi\Pi xp^2 = \frac{\Pi\Pi xpi \,.\, xpj}{(xij)^{2m^2}\,\Pi\Pi(pij)^2} = \frac{(\bar{xi}^m c_m \,.\, \bar{xj}^m c_m)^m\,(\bar{ij}^m c_m)^2}{(xij)^{2m^2}\,(ij^m c_m)^{2(m+1)}}$$

$$= \frac{(\bar{xi}^m c_m \,.\, \bar{xj}^m c_m)^m}{(xij)^{2m^2}\,(ij^m c_m)^{2m}},$$

and therefore

$$\Pi xp^2 = \frac{\bar{xi}^m c_m \,.\, \bar{xj}^m c_m}{(xij)^{2m}\,(\bar{ij}^m c_m)^2}.$$

The curve C_n has n asymptotes, which are the tangents at the points where it is met by the line ij. If a point x lie on one of the asymptotes, its first polar xC_n must meet C_n on the line ij. The condition for this is of the order n in xC_n, $n-1$ in C_n, and $n(n-1)$ in ij; that is, of the order n in x, $2n-1$ in C_n, and $n(n-1)$ in ij. It must also be of the form

$$Ax^n C_n + x^{n-2} B_{n-2} \,.\, (xij)^2 = 0,$$

since the n asymptotes form a curve of the nth order touching C_n where it is met by ij. If A vanishes, the line ij becomes a double factor; now this can only happen when $\bar{ij}^m c_m = 0$, and a comparison of dimensions shews that A differs from this only by a numerical factor. We may therefore write for the equation of the asymptotes

$$\bar{ij}^m c_m \,.\, C_n + B_{n-2} \,.\, \bar{ij}^2 \equiv \Pi P.$$

We may now find the product of the sines of the angles between them, $\Pi \sin PQ$, and of the angles they make with a line X, $\Pi \sin XP$. Namely, we have

$$\Pi \sin^2 PQ = \frac{\Pi\,(PQij)^2}{(\Pi Pi \,.\, \Pi Pj)^{n-1}} \quad [\text{to factor } \textit{près}].$$

Now

$$(\Pi PQij)^2 = (\bar{ij}^m c_m)^{2n-1},$$

and therefore

$$\Pi \sin^2 PQ = \frac{(\bar{ij}^m c_m)^{2n-1}}{(\bar{ij}^m c_m)^{2(n-1)}\,(i^2 C_n \,.\, j^2 C_n)^{n-1}} = \frac{\bar{ij}^m c_m}{(i^2 C_n \,.\, j^2 C_n)^{n-1}}.$$

In the next place

$$\Pi \sin^2 XP = \frac{\Pi\,(XPij)^2}{(Xi \,.\, Xj)^n \,.\, \Pi Pi \,.\, \Pi Pj} = \frac{\{(Xij)^n C_n\}^2}{(Xi \,.\, Xj)^n \,.\, i^n C_n \,.\, j^n C_n}.$$

Let us now suppose a variable line X to be drawn through the fixed point x, meeting the curve C_n in the points l, m, n,... then

$$\Pi xl^2 = \frac{\Pi xli \,.\, \Pi xlj}{(xij)^{2n}\,\Pi\,(lij)^2} = \frac{(x^n C_n)^2 \,.\, (Xi)^n \,.\, (Xj)^n}{(xij)^{2n} \,.\, \{(Xij)^n C_n\}^2},$$

or

$$\Pi xl = \frac{x^n C_n \,.\, (Xi \,.\, Xj)^{\frac{n}{2}}}{(xij)^n \,.\, (Xij)^n C_n}.$$

Hence

$$\Pi xl \,.\, \Pi \sin XP = \frac{x^n C_n}{(xij)^n \,.\, \sqrt{(i^n C_n \,.\, j^n C_n)}};$$

this product is therefore independent of the position of the line X, and may be called the *distance* of the point x from the curve C_n.

If we draw to the curve from the point x the tangents $L, M, N, \ldots$ we shall have

$$\Pi \sin^2 LM = \frac{\Pi (LMij)^2}{(\Pi Li \,.\, \Pi Lj)^{m-1}} = \frac{x^n C_n \,.\, (xij)^n}{(\overline{xi}^m c_m \,.\, \overline{xj}^m c_m)^{m-1}};$$

$$\therefore (\Pi xp^2)^{m-1} \,.\, \Pi \sin^2 LM = \frac{x^n C_n}{(xij)^n \,.\, (\overline{ij}^2 c_m)^{2(m-1)}},$$

and finally

$$\frac{(\Pi xp^2)^{m-1}}{\Pi pq^2} \,.\, \Pi \sin^2 LM = \frac{x^n C_n}{(xij)^n \,.\, \sqrt{(i^n C_n \,.\, j^n C_n)}} \text{ [to a factor]}$$

$$= \Pi xl \,.\, \Pi \sin XP \text{ to a factor,}$$

and by supposing the curve to break up into m points the factor is easily determined to be unity.

Considering now the line X as fixed and the point x as variable, we have

$$\Pi \sin^2 XL = \frac{\Pi (XLij)^2}{(Xi \,.\, Xj)^m \,\Pi Li \,.\, \Pi Lj} = \frac{(X^m c_m)^2 \,(xij)^{2m}}{(Xi \,.\, Xj)^m \,.\, \overline{xi}^m c_m \,.\, \overline{xj}^m c_m};$$

$$\therefore \Pi \sin XL \,.\, \Pi xp = \frac{X^m c_m}{(Xi \,.\, Xj)^{\frac{m}{2}} \,.\, \overline{ij}^m c_m}.$$

But also

$$\Pi lm^2 = \frac{\Pi lmi \,.\, \Pi lmj}{(\Pi lij)^{2(n-1)}} = \frac{X^m c_m \,.\, (Xi \,.\, Xj)^{\frac{m}{2}}}{\{(Xij)^n C_n\}^{2(n-1)}},$$

and therefore

$$(\Pi \sin^2 XP)^{n-1} \,.\, \Pi lm^2 = \frac{X^m c_m}{(Xi \,.\, Xj)^{\frac{m}{2}} \,(i^n C_n \,.\, j^n C_n)^{n-1}},$$

whence

$$\frac{(\Pi \sin^2 XP)^{n-1}}{\Pi \sin^2 PQ} \,\Pi lm^2 = \frac{X^m c_m}{(Xi \,.\, Xj)^{\frac{m}{2}} \,.\, \overline{ij}^m c_m}$$

$$= \Pi \sin XL \,.\, \Pi xp, \text{ to a factor,}$$

which, as before, the special case of n lines shews to be unity. We shall call the quantity for which three expressions are here given the *distance* of the line X from the curve c_m.

V.

The elliptic geometry of two dimensions as Dr Klein calls it, or, which is the same thing, geometry on the sphere in which two opposite points are regarded as identical, differs from plane geometry in that instead of the two points ij we have the proper conic O_2 or o_2. Lines touching this conic, and points lying on it, are at an infinite distance from all other lines and points; distances measured on them are zero. Using the ordinary co-ordinates we may write

$$O_2 = x_1^2 + x_2^2 + x_3^2,$$
$$o_2 = \xi_1^2 + \xi_2^2 + \xi_3^2,$$
$$a = a_1\xi_1 + a_2\xi_2 + a_3\xi_3,$$
$$A = A_1x_1 + A_2x_2 + A_3x_3,$$

and then

$$\sin^2 ab = \frac{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}{(a_1^2 + a_2^2 + a_3^2)\,(b_1^2 + b_2^2 + b_3^2)}$$
$$= \frac{2\overline{ab}^2 o_2}{a^2O_2 \,.\, b^2O_2} \text{ [when } O_2 \text{ given]},$$
$$\sin^2 AB = \frac{(A_2B_3 - A_3B_2)^2 + (A_3B_1 - A_1B_3)^2 + (A_1B_2 - A_2B_1)^2}{(A_1^2 + A_2^2 + A_3^2)\,(B_1^2 + B_2^2 + B_3^2)}$$
$$= \frac{2\overline{AB}^2 O_2}{A^2o_2 \,.\, B^2o_2}.$$

The distance of a point a from a line B may be derived from these two as follows (fig. 17): through a draw a variable line A meeting B in b; then we have

$$\sin^2 ab = \sin^2 aAB = \frac{2\,(aB)^2 \,.\, A^2o_2}{a^2O_2 \,.\, \overline{AB}^2 O_2},$$

and
$$\sin^2 AB = \frac{2\overline{AB}^2 O_2}{A^2o_2 \,.\, B^2o_2};$$

$$\therefore \sin^2 ab \,.\, \sin^2 AB = \frac{4\,(aB)^2}{a^2O_2 \,.\, B^2o_2};$$

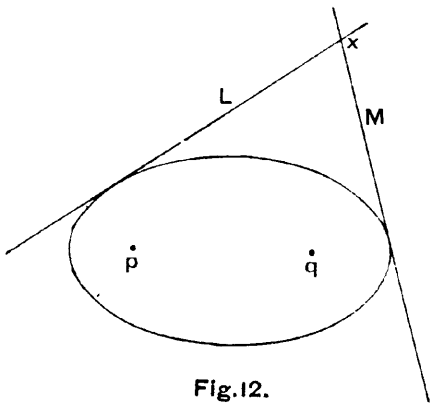


Fig.12.

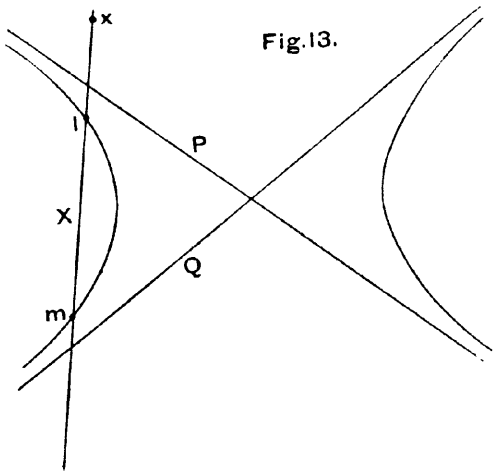


Fig.13.

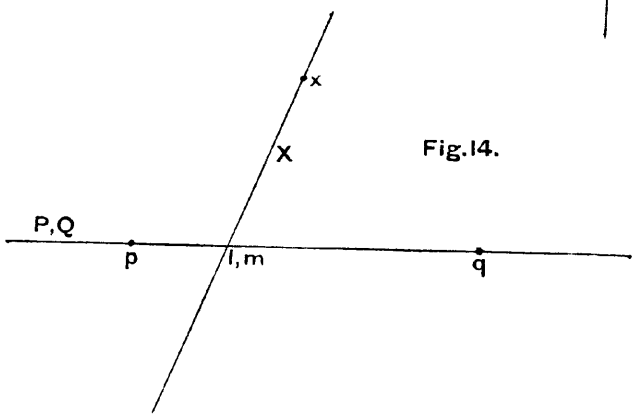
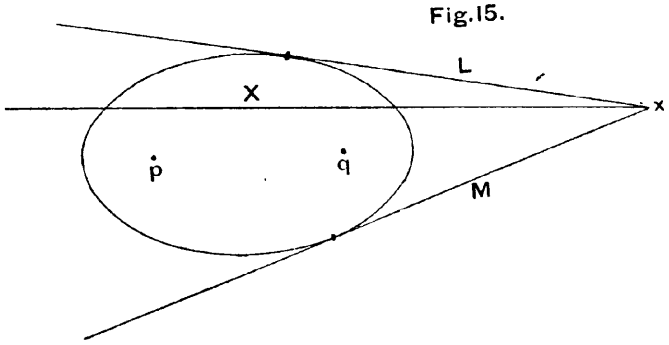


Fig.14.

Fig.15.



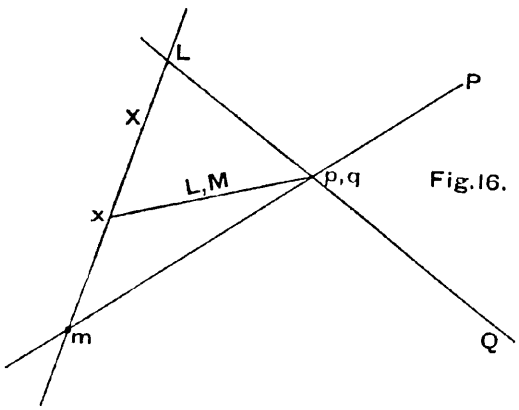
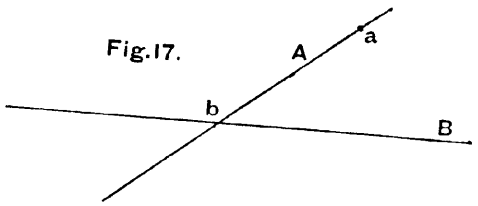


Fig.16.

Fig.17.



face p. 152.

a result independent of the position of the line A, which when A is at right angles to B becomes $\sin^2 aB$.

A conic C_2 or c_2 has with the absolute O_2 or o_2 four common points and four common tangents. The lines joining the common points form three pairs of common chords, P, Q; P', Q'; P'', Q''; the intersections of the common tangents form three pairs of foci, p, q; p', q'; p'', q''. The theories of the foci and the common chords are identical, and it will be sufficient to consider one of them; we shall choose for this purpose the foci.

We obtain a pair of foci by determining λ so that

$$c_2 + \lambda o_2$$

shall break up into factors; the condition for this is

$$0 = 6 \text{ Disct. } (c_2 + \lambda o_2) = C_2 c_2 + 3\lambda C_2 o_2 + 3\lambda^2 c_2 O_2 + \lambda^3 O_2 o_2,$$

and substituting here for λ, $-\frac{c_2}{o_2}$, we obtain the equation of the three pairs of foci:—

$$0 = C_2 c_2 . o_2^3 - 3 C_2 o_2 . o_2^2 . c_2 + 3 c_2 O_2 . o_2 . c_2^2 - O_2 o_2 . c_2^3.$$

If $c_2 + \lambda o_2$ break up into factors p, q, its reciprocal will be $-\frac{1}{4}\overline{pq}^2$. Thus we have

$$-\tfrac{1}{4}\overline{pq}^2 = C_2 + 2\lambda (co)_2 + \lambda^2 O_2,$$

where $(co)_2$ is the locus of points at which c_2 subtends a right angle. Consequently

$$-\tfrac{1}{4}\overline{pq}^2 o_2 = C_2 o_2 + 2\lambda c_2 O_2 + \lambda^2 O_2 o_2 = 2\partial_\lambda \text{ Disct. } (c_2 + \lambda o_2).$$

Moreover

$$pq O_2 = c_2 O_2 + \lambda O_2 o_2 = \partial_\lambda^2 \text{ Disct. } (c_2 + \lambda o_2),$$

but

$$p^2 O_2 . q^2 O_2 - (pq O_2)^2 = \tfrac{1}{3}\overline{pq}^2 o_2 . O_2 o_2,$$

hence

$$\begin{aligned} 3 p^2 O_2 . q^2 O_2 &= 3 (c_2 O_2 + \lambda O_2 o_2)^2 - 4 (C_2 o_2 + 2\lambda c_2 O_2 + \lambda^2 O_2 o_2) O_2 o_2 \\ &= 3 \overline{c_2 O_2}^2 - 4 C_2 o_2 . O_2 o_2 - 2\lambda c_2 O_2 . O_2 o_2 - \lambda^2 \overline{O_2 o_2}^2 \\ &= 3 (\overline{c_2 O_2}^2 - C_2 o_2 . O_2 o_2) - (C_2 o_2 + 2\lambda c_2 O_2 + \lambda^2 O_2 o_2) O_2 o_2. \end{aligned}$$

To simplify these formulæ, let us write

$$6\ \text{Disct.}\ (c_2 + \lambda o_2) = f = a\lambda^3 + 3b\lambda^2 + 3c\lambda + d$$

$$\tfrac{1}{3}\partial_\lambda f = x = a\lambda^2 + 2b\lambda + c$$

$$y = 3\,(b^2 - ac) - ax\,;$$

then whenever λ has a value which makes f vanish, so that $c_2 + \lambda o_2$ breaks up into the factors p, q, we must have

$$-\tfrac{1}{4}\,\overline{pq}^2 o_2 = x,$$

$$3p^2 O_2 \cdot q^2 O_2 = y.$$

We shall now prove that if $x_1\ x_2\ x_3$ are the values of x, and $y_1\ y_2\ y_3$ the values of y, corresponding to the three values of λ given by $f = 0$, then

$$x_1^2 y_1 = x_2^2 y_2 = x_3^2 y_3 = -R_f, \text{ the discriminant of } f.$$

We get an equation for determining x by eliminating λ between $f = 0$ and $\tfrac{1}{3}\partial_\lambda f - x = 0$. Namely, the resultant of these two equations is

$$\begin{vmatrix} a & 3b & 3c & d & . \\ . & a & 3b & 3c & d \\ a & 2b & c-x & . & . \\ . & a & 2b & c-x & . \\ . & . & a & 2b & c-x \end{vmatrix} = -a^2x^3 + 3\,(b^2 - ac)\,ax^2 + aR_f,$$

$$= a\,(R_f + x^2 y),$$

which vanishes if

$$x^2 y = -R_f;$$

this equation is therefore true for each of the corresponding pairs of values of x and y. Substituting for these their values, we have

$$(\overline{pq}^2 o_2)^2 \cdot p^2 O_2 \cdot q^2 O_2 = (\overline{p'q'}^2 o_2)^2 \cdot p'^2 O_2 \cdot q'^2 O_2$$

$$= (\overline{p''q''}^2 o_2)^2 \cdot p'^2 O_2 \cdot q''^2 O_2 = -\tfrac{16}{3} R_f.$$

Now R_f is the osculant of c_2 and o_2, that is, the invariant whose vanishing is the condition that the two conics shall touch.

It follows further from the cubic equation for x that the product of its three values is $\frac{R_f}{a}$. Hence we have

$$\overline{pq}^2 o_2 \cdot \overline{p'q'}^2 o_2 \cdot \overline{p''q''}^2 o_2 = -64\frac{R_f}{a},$$

and therefore

$$p^2O_2 \cdot q^2O_2 \cdot p'^2O_2 \cdot q'^2O_2 \cdot p''^2O_2 \cdot q''^2O_2 = -\frac{a^2R_f}{27};$$

whence by division

$$\sin^2 pq \cdot \sin^2 p'q' \cdot \sin^2 p''q'' = +\frac{12^3[2^3]}{6^3} = +[64],*$$

since $$\sin^2 pq = \frac{[2]\overline{pq}^2 o_2}{p^2O_2 \cdot q^2O_2} \cdot \frac{O_2o_2}{6}$$ when o_2 is given.

From the same equation we learn that the sum of the reciprocals of the x is zero. Now

$$-\frac{R_f}{x^3} = \frac{y}{x} = [-2O_2o_2](\sin pq)^{-2};$$

therefore

$$(\sin pq)^{-\frac{2}{3}} + (\sin p'q')^{-\frac{2}{3}} + (\sin p''q'')^{-\frac{2}{3}} = 0.$$

$$[\text{Also } (\sin pq)^{-2} + (\sin p'q')^{-2} + (\sin p''q'')^{-2} = \tfrac{3}{4}.]*$$

We have thus two equations connecting the quantities $\sin pq$, $\sin p'q'$, $\sin p''q''$, by which when one is given the other two may be determined.

Now let x be any point of the plane (or sphere). Then we have

$$\sin^2 xp \cdot \sin^2 xq = \frac{\overline{xp}^2 o_2 \cdot \overline{xq}^2 o_2}{(x^2O_2)^2 \cdot p^2O_2 \cdot q^2O_2} \left[\left(\frac{O_2o_2}{6}\right)^2\right].$$

But the numerator of this fraction is clearly the result of operating with x^4 on the common tangents of pq and o_2, that is of $c_2 + \lambda o_2$ and o_2, a result which is clearly independent of λ. Hence we have

$$\overline{xp}^2 o_2 \cdot \overline{xq}^2 o_2 = \overline{xp'}^2 o_2 \cdot \overline{xq'}^2 o_2 = \overline{xp''}^2 o_2 \cdot \overline{xq''}^2 o_2.$$

Moreover, since

$$(\overline{pq}^2 o_2)^2 p^2O_2 \cdot q^2O_2 = -\tfrac{16}{3} R_f,$$

* [The introduced factor, cf. p. 152, brings the results into accordance with Prof. Cayley's equations. See Note to this paper, p. 159 (3).]

it follows that

$$\sin^4 pq = \tfrac{1}{36}\left(\frac{\overline{pq}^2 o_2 \cdot O_2 o_2}{p^2 O_2 \cdot q^2 O_2}\right)^2 = -\tfrac{4}{27}\frac{R_f (O_2 o_2)^2}{(p^2 O_2 \cdot q^2 O_2)^3},$$

and consequently

$$\frac{\sin xp \cdot \sin xq}{(\sin pq)^{\frac{2}{3}}} = \frac{\sin xp' \cdot \sin xq'}{(\sin p'q')^{\frac{2}{3}}} = \frac{\sin xp'' \cdot \sin xq''}{(\sin p''q'')^{\frac{2}{3}}}$$

$$\left[= \frac{\sqrt{\overline{xp}^2 o_2 \cdot \overline{xq}^2 o_2}}{x^2 O_2} \frac{1}{\sqrt[3]{16}\sqrt{3}} \frac{(O_2 o_2)^{\frac{2}{3}}}{\sqrt[6]{-R_f}}\right]$$

$$= \left[\frac{1}{\sqrt[3]{16}\sqrt{3}}\right] \frac{\sqrt{x^4 (co)_4} (O_2 o_2)^{\frac{2}{3}}}{x^2 O_2 \sqrt[6]{-R_f}}.$$

The three equations which we have just proved establish the theory of three pairs of antipoints on a sphere; viz. the three pairs of intersections of four tangents to the absolute. They take the place of the *two* equations which we have already used in regard to the two pairs of antipoints on a plane; namely,

$$pq^2 = -p'q'^2,$$

and

$$xp \cdot xq = xp' \cdot xq'.$$

We now proceed to use them in connection with the theory of the conic c_2.

From the point x let the tangents L, M be drawn to the conic; then

$$\sin^2 LM = [2]\frac{\overline{LM}^2 O_2}{\overline{L}^2 o_2 \cdot \overline{M}^2 o_2} = \frac{x^2 O_2 \cdot x^2 C_2}{\overline{xp}^2 o_2 \cdot \overline{xq}^2 o_2} \text{ [to factor } près\text{]},$$

where pq are any set of foci. But also

$$\frac{\sin^2 xp \cdot \sin^2 xq}{(\sin^2 pq)^{\frac{2}{3}}} = \left[\frac{1}{12\sqrt[3]{4}}\right] \cdot \frac{\overline{xp}^2 o_2 \cdot \overline{xq}^2 o_2 (O_2 o_2)^{\frac{1}{3}}}{(x^2 O_2)^2 \sqrt[3]{-R_f}},$$

and therefore

$$\frac{\sin^2 xp \cdot \sin^2 xq}{(\sin^2 pq)^{\frac{2}{3}}} \sin^2 LM = \left[\frac{1}{12\sqrt[3]{4}}\right] \frac{x^2 C_2 \cdot (O_2 o_2)^{\frac{1}{3}}}{x^2 O_2 \cdot \sqrt[3]{-R_f}}$$

$$= \left[\frac{1}{12\sqrt[3]{4}}\right] \cdot \frac{x^2 C_2 \cdot (O_2 o_2)^2 (C_2 c_2)^{\frac{2}{3}}}{x^2 O_2 \cdot \sqrt[3]{-R_F}}.$$

Again, if we draw through x a variable line X, meeting the conic in l, m, we shall have

$$\sin^2 xl \,.\, \sin^2 xm = \frac{\overline{xl^2}o_2 \,.\, \overline{xm^2}o_2}{(x^2O_2)^2 \,.\, l^2O_2 \,.\, m^2O_2} = \frac{(x^2C_2)^2 \,.\, (X^2o_2)^2}{(x^2O_2)^2 \,.\, X^4(CO)_4}$$

$$= \frac{(x^2C_2)^2 \,.\, (X^2o_2)^2}{(x^2O_2)^2 \,.\, \overline{XP^2}O_2 \,.\, \overline{XQ^2}O_2},$$

$$\text{and } \frac{\sin^2 XP \,.\, \sin^2 XQ}{(\sin^2 PQ)^{\frac{2}{3}}} = \left[\frac{1}{6\frac{4}{3}}\right] \frac{\overline{XP^2}O_2 \,.\, \overline{XQ^2}O_2\,(O_2o_2)^{\frac{1}{3}}}{(X^2o_2)^2\,\{P^2o_2 \,.\, Q^2o_2\,(PQ^2O_2)^2\}^{\frac{1}{3}}}$$

$$= \tfrac{3}{16} \frac{\overline{XP^2}O_2 \,.\, \overline{XQ^2}O_2 \,.\, (O_2o_2)^{\frac{2}{3}}}{(X^2o_2)^2 \sqrt[3]{R_F}},$$

where R_F signifies the osculant of C_2 and O_2; $R_F = (C_2c_2)^2 (O_2o_2)^2 R_f$.

Therefore

$$\sin xl \,.\, \sin xm \,\frac{\sin XP \sin XQ}{(\sin PQ)^{\frac{2}{3}}} = \left[\frac{1}{\sqrt[6]{12^3 \,.\, 4}}\right] \frac{x^2C_2 \,.\, (O_2o_2)^{\frac{2}{3}}}{x^2O_2 \,.\, \sqrt[6]{-R_F}}.$$

[The MS. here ends abruptly. In the results of the last two pages I have introduced a few additions, all, or nearly all, of which agree with results obtained by Prof. Henrici, who most kindly gave me his valuable help in revising the proof sheets.

The subject of V. has been treated from a different point of view by Prof. Cayley, who has allowed me to append the following note.

The results obtained in No. V. of the paper On the Theory of Distances may be worked out in greater detail, and in some measure in a more complete form.

Using line-coordinates, we have $x^2+y^2+z^2=0$, the conic called the absolute; a conic $(a, b, c, f, g, h)(x, y, z)^2=0$, which will be called simply the conic; and a point (x) the equation of which is $lx+my+nz=0$. The common tangents of the conic and the absolute intersect in pairs in six points $p, q : p', q' : p'', q''$, which are the foci of the conic; or if we regard the four lines simply as any four tangents of the absolute, then the six points are a system of foci; and we obtain in the first instance formulæ relating to such a system, alone or in connection with the point x: afterwards, taking them to be the foci of the conic, we further consider the two tangents L, M from the point (x) to the conic; and an arbitrary line X through the point (x).

1. The coordinates of a tangent of the absolute are (x_1, y_1, z_1), where these are any values such that $x_1^2+y_1^2+z_1^2=0$; and we consider the four tangents

$$(x_1, y_1, z_1), \quad (x_2, y_2, z_2), \quad (x_3, y_3, z_3), \quad (x_4, y_4, z_4).$$

Writing for a moment

$$\lambda,\ \mu,\ \nu = y_1z_4 - y_4z_1,\quad z_1x_4 - z_4x_1,\quad x_1y_4 - x_4y_1,$$

$$\lambda',\ \mu',\ \nu' = y_2z_3 - y_3z_2,\quad z_2x_3 - z_3x_2,\quad x_2y_3 - x_3y_2,$$

we have $$\lambda x + \mu y + \nu z = 0,\quad \lambda' x + \mu' y + \nu' z = 0$$

for the equations of the points p and q respectively; and the expression for the distance is given by

$$\cos^2 pq = \frac{(\lambda\lambda' + \mu\mu' + \nu\nu')^2}{(\lambda^2 + \mu^2 + \nu^2)(\lambda'^2 + \mu'^2 + \nu'^2)};$$

and if for shortness we write

$$12 = x_1x_2 + y_1y_2 + z_1z_2,\ \&c.,$$

then the values of λ, μ, ν give

$$\lambda^2 + \mu^2 + \nu^2 = -(14)^2,$$

$$\lambda'^2 + \mu'^2 + \nu'^2 = -(23)^2,$$

$$\lambda\lambda' + \mu\mu' + \nu\nu' = -31\,.\,24 + 12\,.\,34.$$

I write $$\sqrt{23\,.\,14} = f,\quad \sqrt{31\,.\,24} = g,\quad \sqrt{12\,.\,34} = h,$$

and I say that we have $f + g + h = 0$. The formula becomes

$$\cos^2 pq = \frac{(g^2 - h^2)^2}{f^4},$$

and we thence have

$$\sin^2 pq = \frac{(f^2 - g^2 + h^2)(f^2 + g^2 - h^2)}{f^4} = \frac{-2fh\,.\,-2fg}{f^4},\quad = \frac{4gh}{f^2};$$

and consequently for the three pairs of foci respectively

$$\sin^2 pq = \frac{4gh}{f^2};\ \sin^2 p'q' = \frac{4hf}{g^2};\ \sin^2 p''q'' = \frac{4fg}{h^2}. \qquad *$$

2. The assumed relation $f + g + h = 0$ is obtained from the equation

$$\begin{vmatrix} x_1, & y_1, & z_1, & w_1 \\ x_2, & y_2, & z_2, & w_2 \\ x_3, & y_3, & z_3, & w_3 \\ x_4, & y_4, & z_4, & w_4 \end{vmatrix}^2 = 0,$$

which is identically true if w_1, w_2, w_3, w_4 are each $= 0$; attending to the equations $x_1^2 + y_1^2 + z_1^2 = 0$, &c., this is

$$\begin{vmatrix} . & , & (12)^2, & (13)^2, & (14)^2 \\ (21)^2, & . & , & (23)^2, & (24)^2 \\ (31)^2, & (32)^2, & . & , & (34)^2 \\ (41)^2, & (42)^2, & (43)^2, & . & \end{vmatrix} = 0,$$

which is in fact the rationalised form of $f + g + h = 0$.

* Results are marked with an asterisk.

3. The foregoing values give

$$\left.\begin{aligned} \sin^2 pq \,.\, \sin^2 p'q' \,.\, \sin^2 p''q'' &= 64, \\ \sin^{-\frac{2}{3}} pq + \sin^{-\frac{2}{3}} p'q' + \sin^{-\frac{2}{3}} p''q'' &= \frac{\mathrm{f}+\mathrm{g}+\mathrm{h}}{(4\mathrm{fgh})^{\frac{2}{3}}}, \quad = 0 \\ \sin^{-2} pq + \sin^{-2} p'q' + \sin^{-2} p''q'' &= \frac{\mathrm{f}^3+\mathrm{g}^3+\mathrm{h}^3}{4\mathrm{fgh}}, \quad = \tfrac{3}{4}. \end{aligned}\right\} \quad *$$

4. Considering now, in connection with the foci, the point (x) determined by the equation $lx+my+nz=0$, we have

$$\cos^2 xp = \frac{(l\lambda+m\mu+n\nu)^2}{(l^2+m^2+n^2)\,(\lambda^2+\mu^2+\nu^2)},$$

λ, μ, ν as before, and therefore

$$\lambda^2+\mu^2+\nu^2 = -(14)^2.$$

Moreover

$$(l\lambda+m\mu+n\nu)^2 = \begin{vmatrix} l, & m, & n \\ x_1, & y_1, & z_1 \\ x_4, & y_4, & z_4 \end{vmatrix}^2, = \begin{vmatrix} l^2+m^2+n^2, & 01, & 04 \\ 01 & , \,.\, , & 14 \\ 04 & , 14, & . \end{vmatrix},$$

(if for shortness $01 = lx_1+my_1+nz_1$, &c.)

$$= -(l^2+m^2+n^2)\,(14)^2 + 2\,.\,01\,.\,04\,.\,14.$$

The formula thus is

$$\cos^2 xp = \frac{-(l^2+m^2+n^2)\,(14)^2+2\,.\,01\,.\,04\,.\,14}{-(l^2+m^2+n^2)\,(14)^2};$$

or passing to $\sin^2 xp$, and then writing down the analogous value of $\sin^2 xq$, we have

$$\sin^2 xp = \frac{2\,.\,01\,.\,04}{14\,.\,l^2+m^2+n^2}; \quad \sin^2 xq = \frac{2\,.\,02\,.\,03}{23\,.\,l^2+m^2+n^2};$$

and in like manner for the other two pairs of foci

$$\sin^2 xp' = \frac{2\,.\,02\,.\,04}{24\,.\,l^2+m^2+n^2}; \quad \sin^2 xq' = \frac{2\,.\,03\,.\,01}{31\,.\,l^2+m^2+n^2},$$

$$\sin^2 xp'' = \frac{2\,.\,03\,.\,04}{34\,.\,l^2+m^2+n^2}; \quad \sin^2 xq'' = \frac{2\,.\,01\,.\,02}{12\,.\,l^2+m^2+n^2}.$$

5. These formulæ give

$$\sin^2 xp \,\sin^2 xq \,\sin^{-\frac{4}{3}} pq = \sin^2 xp' \,\sin^2 xq' \,\sin^{-\frac{4}{3}} p'q' = \sin^2 xp'' \,\sin^2 xq'' \,\sin^{-\frac{4}{3}} p''q''$$

$$= 4\,.\,01\,.\,02\,.\,03\,.\,04\,.\,(4\mathrm{fgh})^{-\frac{2}{3}}\,.\,(l^2+m^2+n^2)^{-2};$$

or as this may also be written

$$= 4^{\frac{1}{3}}\,(01\,.\,02\,.\,03\,.\,04)\,(12\,.\,13\,.\,14\,.\,23\,.\,24\,.\,34)^{-\frac{1}{3}}\,(l^2+m^2+n^2)^{-2}, \quad *$$

where it will be recollected that 01 denotes $lx_1+my_1+nz_1$, &c., and 12 denotes $x_1x_2+y_1y_2+z_1z_2$, &c....

6. Taking now (x_1, y_1, z_1), &c., as the common tangents of the absolute and the conic, or say as the roots of the equations

$$x^2+y^2+z^2=0, \quad (a, b, c, f, g, h)(x, y, z)^2=0,$$

the expression on the right hand side, qua symmetrical function, homogeneous of the degree zero in the roots, and also homogeneous of the degree zero in the coefficients l, m, n, will be expressible as an absolute invariant of the two quadric functions and of the linear function $lx+my+nz$: and I say that the value is

$$= -4^{-\frac{2}{3}} . \square . \Omega^{-\frac{1}{3}} (l^2+m^2+n^2)^{-2},$$

$\square$ being the Resultant of the three functions, and Ω the Tactinvariant of the two quadric functions, as presently appearing. It is to be observed that $l^2+m^2+n^2$ is in fact the Reciprocant of $lx+my+nz$ and $x^2+y^2+z^2$, viz. the Reciprocant of $(a, \ldots)(x, y, z)^2$ and $lx+my+nz$ is

$$(bc-f^2, ca-g^2, ab-h^2, gh-af, hf-bg, fg-ch)(l, m, n)^2,$$

and for the quadric function $x^2+y^2+z^2$ this becomes $=l^2+m^2+n^2$.

7. Considering the three functions

$$x^2+y^2+z^2,$$
$$(a, b, c, f, g, h)(x, y, z)^2,$$
$$lx+my+nz,$$

it will be sufficient as regards the resultant to write down those terms which are independent of f, g, h; these are at once obtained by writing f, g, h each $=0$, and the resultant then presents itself as the norm of $l\sqrt{b-c}+m\sqrt{c-a}+n\sqrt{a-b}$; and we thus obtain (attending only to the terms in question)

$$\square = l^4(b-c)^2+m^4(c-a)^2+n^4(a-b)^2$$
$$-2m^2n^2(c-a)(a-b)-2n^2l^2(a-b)(b-c)-2l^2m^2(b-c)(c-a).$$

The Resultant is at once expressed in terms of the roots (x_1, y_1, z_1), &c., by the formula

$$\square = C(lx_1+my_1+nz_1)(lx_2+my_2+nz_2)(lx_3+my_3+nz_3)(lx_4+my_4+nz_4),$$

or according to the foregoing notation

$$\square = C . 01 . 02 . 03 . 04,$$

where, and in what follows, C *is written to denote an essentially indeterminate constant*, having (it may be) different values in different equations.

8. Moreover writing as usual

$$K = abc-af^2-bg^2-ch^2+2fgh,$$
$$\Theta = bc-f^2+ca-g^2+ab-h^2,$$
$$\Theta' = a+b+c,$$
$$K' = 1,$$

the Tactinvariant is taken to be

$$\Omega = 27K^2K'^2 + 4K\Theta'^3 + 4K'\Theta^3 - 18KK'\Theta\Theta' - \Theta^2\Theta'^2$$

(which for f, g, h each $=0$, reduces itself to

$$\Omega = -(b-c)^2(c-a)^2(a-b)^2).$$

The Tactinvariant vanishes if, and only if, a pair of roots (x_1, y_1, z_1), (x_2, y_2, z_2) become identical, say $x_1 : y_1 : z_1 = x_2 : y_2 : z_2$. But we have $(y_1z_2 - y_2z_1)^2 = (y_1^2 + z_1^2)(y_2^2 + z_2^2) - (y_1y_2 + z_1z_2)^2, = 0$, if $x_1x_2 + y_1y_2 + z_1z_2 = 0$, that is if $12 = 0$; and similarly $(z_1x_2 - z_2x_1)^2 = 0$, and $(x_1y_2 - x_2y_1)^2 = 0$, if $12 = 0$. And we are thus led to the equation

$$\Omega = C.12.13.14.23.24.34.$$

9. The combination $\square^3\Omega^{-1}$ contains the roots homogeneously in the degree zero, and it will therefore have a determinate value, which is in fact found by the process which I present as a verification. The result is

$$\square^3\Omega^{-1} = -64\,(01.02.03.04)^3\,(12.13.14.23.24.34)^{-1}.$$

In verification, take the function $(a, \ldots \mathcal{X}x, y, z)^2$ to be $= x^2 + \omega y^2 + \omega^2 z^2$, ω an imaginary cube root of unity: the roots may be taken to be $(1, \omega^2, \omega)$, $(1, -\omega^2, \omega)$, $(1, \omega^2, -\omega)$, $(1, -\omega^2, -\omega)$. Attending only to the terms in l, we have

$$\square = -3l^4;\ \Omega = 27;\ 01.02.03.04 = l^4;\ 12.13.14.23.24.14$$

(is a product of factors such as $1 - \omega + \omega^2, = -2\omega$, and is) $= -64$; or the equation becomes $(-3l^4)^3(27)^{-1} = -64l'^2(64)^{-1}$, which is right. We have thus

$$-4\square\,\Omega^{-\frac{1}{3}} = (01.02.03.04)\,(12.13.14.23.24.34)^{-\frac{1}{3}},$$

and hence the foregoing value of $\sin^2 xp\,\sin^2 xq\,\sin^{-\frac{4}{3}} pq$: say

$$\sin^2 xp\,\sin^2 xq\,\sin^{-\frac{4}{3}} pq = -4^{-\frac{2}{3}}\square\,\Omega^{-\frac{1}{3}}(l^2 + m^2 + n^2)^{-2}. \quad *$$

10. From the point (x) we draw to the conic tangents L, M: taking their coordinates to be (x_1, y_1, z_1), (x_2, y_2, z_2), these are the roots of

$$lx + my + nz = 0,$$
$$(a, \ldots \mathcal{X}x, y, z)^2 = 0,$$

and we have

$$\sin^2 LM = \frac{(x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2) - (x_1x_2 + y_1y_2 + z_1z_2)^2}{(x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2)}.$$

11. We have $x_1^2 + y_1^2 + z_1^2 = 0$, or $x_2^2 + y_2^2 + z_2^2 = 0$, as the condition in order that the resultant $\square$ may vanish, and consequently

$$\square = C\,(x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2).$$

It is easy to see that the function in the numerator will vanish if $l^2 + m^2 + n^2 = 0$, or if the Reciprocant $(bc - f^2, \ldots \mathcal{X}l, m, n)^2$ of the function $(a, \ldots \mathcal{X}x, y, z)^2$ and $lx + my + nz$ is $= 0$: or calling this reciprocant F we have

$$(l^2 + m^2 + n^2)F = C\{(x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2) - (x_1x_2 + y_1y_2 + z_1z_2)^2\}.$$

The values of C in these two equations have a determinate ratio, and we find

$$\frac{(x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2) - (x_1x_2 + y_1y_2 + z_1z_2)^2}{(x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2)} = \frac{-4F.(l^2 + m^2 + n^2)}{\square}.$$

In verification, assume $(a,\ldots)(x, y, z)^2 = x^2 + \omega y^2 + \omega^2 z^2$ as before, $lx+my+nz = x-z$; the roots (x_1, y_1, z_1) and (x_2, y_2, z_2) may be taken to be (1, 1, 1), (1, −1, 1): we have $\square = \{l^2(b-c) - n^2(a-b)\}^2, = (2\omega - 1 - \omega^2)^2, = 9\omega^2$; $F = bcl^2 + abn^2$, $= b(a+c), = \omega(1+\omega^2), = -\omega^2$, and the equation is

$$\frac{8}{9} = \frac{-4 \,.\, -\omega^2 \,.\, 2}{9\omega^2}.$$

Hence we have

$$\sin^2 LM = -4 \,.\, F\,\square^{-1} \,.\, (l^2+m^2+n^2), \qquad *$$

and consequently also

$$\sin^2 LM \sin^2 xp \sin^2 xq \sin^{-\frac{4}{3}} pq = 4\,3^{\frac{1}{3}}\Omega^{-\frac{1}{3}} F \,.\, (l^2+m^2+n^2)^{-1}, \qquad *$$

which is Clifford's formula, p. 156.

12. We take through the point (x) an arbitrary line X, coordinates (α, β, γ): these coordinates satisfy therefore the equation $\alpha l + \beta m + \gamma n = 0$.

We have

$$\sin^2 XL \,.\, \sin^2 XM$$

$$= \frac{(\alpha^2+\beta^2+\gamma^2)(x_1^2+y_1^2+z_1^2) - (\alpha x_1 + \beta y_1 + \gamma z_1)^2}{(\alpha^2+\beta^2+\gamma^2)(x_1^2+y_1^2+z_1^2)} \cdot \frac{(\alpha^2+\beta^2+\gamma^2)(x_2^2+y_2^2+z_2^2) - (\alpha x_2 + \beta y_2 + \gamma z_2)^2}{(\alpha^2+\beta^2+\gamma^2)(x_2^2+y_2^2+z_2^2)}.$$

13. To reduce this expression write for shortness

$$\alpha^2+\beta^2+\gamma^2 = V,$$
$$x_1^2+y_1^2+z_1^2 = \rho_1,$$
$$x_2^2+y_2^2+z_2^2 = \rho_2,$$
$$\alpha x_1 + \beta y_1 + \gamma z_1 = \sigma_1,$$
$$\alpha x_2 + \beta y_2 + \gamma z_2 = \sigma_2,$$
$$x_1x_2 + y_1y_2 + z_1z_2 = \tau.$$

The expression is

$$\frac{V\rho_1 - \sigma_1^2}{V\rho_1} \cdot \frac{V\rho_2 - \sigma_2^2}{V\rho_2},$$

where the numerator is

$$= V(V\rho_1\rho_2 - \sigma_1^2\rho_2 - \sigma_2^2\rho_1) + \sigma_1^2\sigma_2^2.$$

But from the equations $lx_1 + my_1 + nz_1 = 0$, $lx_2 + my_2 + nz_2 = 0$, $l\alpha + m\beta + n\gamma = 0$, we have

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix} = 0,$$

or squaring and reducing

$$\begin{vmatrix} V, & \sigma_1, & \sigma_2 \\ \sigma_1, & \rho_1, & \tau \\ \sigma_2, & \tau, & \rho_2 \end{vmatrix} = 0,$$

that is

$$V(\rho_1\rho_2 - \tau^2) + 2\sigma_1\sigma_2\tau - \sigma_1^2\rho_2 - \sigma_2^2\rho_1 = 0,$$

and by reason hereof the foregoing numerator becomes

$$V(V\tau^2 - 2\sigma_1\sigma_2\tau) + \sigma_1^2\sigma_2^2, = (V\tau - \sigma_1\sigma_2)^2.$$

We thus have

$$\sin^2 XL \,.\, \sin^2 XM = \frac{(V\tau - \sigma_1\sigma_2)^2}{V^2\rho_1\rho_2},$$

$$= \frac{\{(\alpha^2+\beta^2+\gamma^2)(x_1x_2+y_1y_2+z_1z_2) - (\alpha x_1+\beta y_1+\gamma z_1)(\alpha x_2+\beta y_2+\gamma z_2)\}^2}{(\alpha^2+\beta^2+\gamma^2)^2 \,.\, (x_1^2+y_1^2+z_1^2)(x_2^2+y_2^2+z_2^2)}.$$

14. The numerator-function

$$(\alpha^2+\beta^2+\gamma^2)(x_1x_2+y_1y_2+z_1z_2) - (\alpha x_1+\beta y_1+\gamma z_1)(\alpha x_2+\beta y_2+\gamma z_2)$$

is $\quad = (\beta z_1 - \gamma y_1)(\beta z_2 - \gamma y_2) + (\gamma x_1 - \alpha z_1)(\gamma x_2 - \alpha z_2) + (\alpha y_1 - \beta x_1)(\alpha y_2 - \beta x_2),$

which vanishes if $\alpha : \beta : \gamma = x_1 : y_1 : z_1$ or $= x_2 : y_2 : z_2$, that is if $(a, \ldots \between \alpha, \beta, \gamma)^2 = 0$. Moreover observing that $l : m : n = \beta z_1 - \gamma y_1 : \gamma x_1 - \alpha z_1 : \alpha y_1 - \beta x_1 = \beta z_2 - \gamma y_2 : \gamma x_2 - \alpha z_2 : \alpha y_2 - \beta x_2$, it also vanishes if $l^2+m^2+n^2=0$: and we hence have

$$C\{(\alpha^2+\beta^2+\gamma^2)(x_1x_2+y_1y_2+z_1z_2) - (\alpha x_1+\beta y_1+\gamma z_1)(\alpha x_2+\beta y_2+\gamma z_2)\}$$
$$= (l^2+m^2+n^2) \,.\, (a, \ldots \between \alpha, \beta, \gamma)^2$$

(viz. this equation is true when $l\alpha + m\beta + n\gamma = 0$: it is a particular case of a more general formula where α, β, γ are arbitrary, and there are on the right hand side terms containing the factor $l\alpha + m\beta + n\gamma$). And we have as before

$$C(x_1^2+y_1^2+z_1^2)(x_2^2+y_2^2+z_2^2) = \square.$$

Squaring each side of the first equation, and dividing by the two sides of the second equation, we obtain a determinate result which is

$$\frac{\{(\alpha^2+\beta^2+\gamma^2)(x_1x_2+y_1y_2+z_1z_2) - (\alpha x_1+\beta y_1+\gamma z_1)(\alpha x_2+\beta y_2+\gamma z_2)\}^2}{(x_1^2+y_1^2+z_1^2)(x_2^2+y_2^2+z_2^2)}$$
$$= \frac{(l^2+m^2+n^2)^2\{(a, \ldots \between \alpha, \beta, \gamma)^2\}^2}{\square};$$

viz. if to verify we assume as before $lx+my+nz = x-z$; and $(a, \ldots \between x, y, z)^2 = x^2 + \omega y^2 + \omega^2 z^2$: consequently

$$\gamma = \alpha \text{ and } (a, \ldots \between \alpha, \beta, \gamma)^2 = \alpha^2 + \omega\beta^2 + \omega^2\alpha^2, = -\omega(\alpha^2 - \beta^2):$$

also $\quad (x_1, y_1, z_1), (x_2, y_2, z_2) = (1, 1, 1), (1, -1, 1), \square = 9\omega^2,$

then the equation becomes

$$\frac{\{2\alpha^2+\beta^2-(2\alpha+\beta)(2\alpha-\beta)\}^2}{9} = \frac{4\{\omega(\beta^2-\alpha^2)\}^2}{9\omega^2},$$

which is right.

15. We hence have

$$\sin^2 XL \sin^2 XM = (l^2+m^2+n^2)^2\{(a, \ldots \between \alpha, \beta, \gamma)^2\}^2 \,.\, \square^{-1} \,.\, (\alpha^2+\beta^2+\gamma^2)^{-2}, \qquad *$$

and thence also

$$\sin^2 XL \sin^2 XM \sin^2 xp \sin^2 xq \sin^{-\frac{4}{3}} pq$$
$$= -4^{-\frac{2}{3}} \,.\, \{(a, \ldots \between \alpha, \beta, \gamma)^2\}^2 \,.\, \Omega^{-\frac{1}{3}} \,.\, (\alpha^2+\beta^2+\gamma^2)^{-2}, \qquad *$$

which is the reciprocal of Clifford's formula, p. 157.]

XVII.

ON A CASE OF EVAPORATION IN THE ORDER OF A RESULTANT*.

A PARTICULAR case of the following Theorem was required in the course of my proof that every rational equation has a root†; but I have thought that the theorem itself (though indeed a mere obvious remark) was worthy of being placed on record, because of the extremely small number of results of this kind that have yet been arrived at, and of their great importance in analysis.

Theorem. Let it be required to eliminate x between two equations homogeneous in x and certain other variables y, z, ..., in which equations, however, x only occurs in virtue of the occurrence of a quantity

$$w = x^\alpha y^\beta z^\gamma \ldots\ldots,$$

where

$$\alpha + \beta + \gamma + \ldots\ldots = \mu;$$

let also m, n be the orders of the equations, and h, k the remainders after division of m, n respectively by μ; then the order of the resultant is

$$= \frac{mn - hk}{\mu}.$$

Demonstration. Suppose that p, q are the quotients of the division of m, n respectively by μ; that is to say, let

$$m = p\mu + h, \quad n = q\mu + k,$$

* [From the *Proceedings of the London Mathematical Society*, Vol. III. Nos. 25, 26, pp. 80—82.]

† [See p. 22, supra.]

then the two equations may be written

$$a_m + a_{m-\mu} \cdot w + a_{m-2\mu} \cdot w^2 + \ldots\ldots + a_h w^p = 0,$$
$$b_n + b_{n-\mu} \cdot w + b_{n-2\mu} \cdot w^2 + \ldots\ldots + b_k w^q = 0,$$

where the suffixes of the several coefficients indicate their orders in the variables $y, z, \ldots\ldots$ Instead of directly eliminating x from these equations, we may eliminate w; and the result of this may be written down at once by Professor Sylvester's dialytic method. It is in fact

$$\begin{vmatrix} a_m & a_{m-\mu} \ldots\ldots\ldots & & \\ . & a_m \ldots\ldots\ldots & & \\ . & . \ldots\ldots\ldots & & \\ & & \ldots\ldots\ldots\ldots & \\ & & \ldots b_k \ldots & \\ & & \ldots b_{k+\mu} & b_k \end{vmatrix}$$

where in the principal diagonal of the determinant the constituent a_m occurs q times, and the constituent b_k occurs p times. The order of the resulting term $(a_m)^q \cdot (b_k)^p$ is $mq + kp$, and this therefore (since the determinant must be homogeneous) is the order of the resultant itself. If we had written the b coefficients before the a coefficients, we should have obtained $np + hq$ as the value of the same quantity. These two values are identical, since by hypothesis

$$q(m-h) = p(n-k) = pq\mu,$$

and therefore $qm + pk = pn + qh = r$, suppose.

Now
$$\begin{aligned} mn &= (p\mu + h)(q\mu + k) \\ &= pq\mu^2 + pk\mu + hq\mu + hk \\ &= \mu r + hk; \end{aligned}$$
$$\therefore\ r = \frac{mn - hk}{\mu},$$

as was to be proved.

The following extension is brought to light by a different method of proving the original theorem.

Let it be required to eliminate $k-1$ variables $x, y, \ldots$ from k equations, homogeneous in these, and certain other vari-

ables, in which equations, however, $x, y, \ldots$ only occur in virtue of the occurrence of $k-1$ quantities $u, v, \ldots$ all of the same order μ; let also $m_1, m_2, \ldots m_k$ be the orders of the equations, and

$$m_i = p_i\ \mu + h_i, \quad h_i < \mu;$$

then the order of the resultant is

$$\Pi p\ (\Sigma \frac{h}{p} + \mu).$$

For the equations may be written

$$a_h (u, v, \ldots)^p + \theta a_{h+\mu} (u, v, \ldots)^{p-1} + \ldots + \theta^p a_{h+p\mu} = 0,$$

where the h, p are to be affected successively with the suffixes 1, 2, ... k, and θ may be considered $= 1$. Now these equations may be regarded as having coefficients of the constant order h, but the weight of every coefficient of θ^r equal to $r\mu$. This being so, the degree of the resultant in the uneliminated variables will be the sum of its order and weight calculated on these suppositions. But its order is $h_1 p_2 p_3 \ldots p_k$, or $\frac{h_1}{p_1} \Pi p$, due to the coefficients of the first equation, $\frac{h_2}{p_2} \Pi p$ due to the coefficients of the second, and so on; while its weight is $\mu \Pi p$. Hence the entire order of the resultant is

$$\Pi p\ (\Sigma \frac{h}{p} + \mu),$$

as stated above.

XVIII.

ON A THEOREM RELATING TO POLYHEDRA, ANALOGOUS TO MR COTTERILL'S THEOREM ON PLANE POLYGONS*.

MR COTTERILL'S theorem, presented last year to the Society, is as follows: For every plane polygon of n vertices there is a curve of class $n-3$ touching all the diagonals; the number of diagonals is such as to exactly determine this curve and no more; and when the curve touches the line at infinity, the area of the polygon is zero.

The proof of this depends essentially upon the fact that if we join the vertices of the polygon to any point in its plane, the area of the polygon is equal to the sum of the triangles so formed, taken of course with their proper signs according to the rule of Möbius.

The analogous theorem in space should therefore apply in the first instance to those solids whose volume can be expressed as the sum of tetrahedra, having one vertex at an arbitrary point of space, and the other three at three vertices of the figure; that is to say, it should apply to solids having *triangular faces.*

For such solids I find accordingly that the analogy is very complete and exact. It is convenient to define a plane containing three vertices but not being a face, as a diagonal plane; and

* [From the *Proceedings of the London Mathematical Society*, Vol. IV. No. 51, pp. 178—185. Mr Cotterill's paper is given, in part, in Vol. IV. No. 49.]

a line joining two vertices but not being an edge, as a diagonal line. This being so, the theorems which I shall prove are the following:

For every polyhedron of n *summits having only triangular faces* (Δ-*faced* n-*acron* CAYLEY) *there is a surface of class* n − 4 *touching all the diagonal planes.*

This surface contains all the diagonal lines.

The diagonal planes and lines are so situated, however, that the conditions of touching the planes and containing the lines are precisely sufficient to determine a surface of class n − 4.

When this surface touches the plane at infinity, the volume of the solid is zero.

To apply these propositions to polyhedra having other than triangular faces, we must consider such polygonal faces as *singularities.* Each of them, in fact, may by a small deformation of the polyhedron be resolved into a certain number of triangles; and we may thus regard a quadrangular face, for example, as the special case of two adjacent triangular faces being in one plane. Thus the quadrangular face *abcd* [fig. 18] may be regarded as produced by coplanarity of the triangles *abd, cbd.* The effect of this is also to unite together the two diagonal planes *abc, adc,* and to make the diagonal line *ac* lie in the face. Thus the surface of class $n-4$ must touch the face *abcd*; but it does not in general contain the lines *ac, bd.* It touches the face at their point of intersection. And, in general, it is not necessary to consider the diagonals of a polygonal face as diagonals of the polyhedron, and they do not in fact lie upon the surface d_{n-4}. But *a polygonal face with* m *vertices is a multiple tangent plane of order* m − 3, *and the curve of contact is Mr Cotterill's curve appertaining to the polygon.*

It is interesting to consider from this point of view the correlative propositions. Just as we have regarded a solid with a given number of summits, or *polyacron,* as having normally or in the most general case only triangular faces, while polygonal faces present themselves as singularities, and polyacra possessing them as degenerate forms; so we must regard a *polyhedron,* or

solid with a given number of faces, as having normally or in general only three-edged summits (tripleural summits, CAYLEY), while summits having a greater number of edges will present themselves as singularities, and polyhedra possessing them as degenerate. Every solid with plane faces, except the tetrahedron, must have singularities of one kind or the other; just as only loci of the second order are general at the same time of their order and of their class.

The proof of these results is as follows. Let $a, b, c, \ldots l, m, n$ be the summits of a Δ-faced polyacron, and p any point in space; let also $X=0$ be the equation to the plane at infinity, and the result of substituting in X the coordinates of any point, as a, be denoted by aX. Now if fgh is a face, and the summits f, g, h, looked at from p, go round the face clockwise, then the expression $\frac{(pfgh)}{pX.fX.gX.hX}$ represents the volume of the tetrahedron $pfgh$ according to the rule of Möbius. (Here $(pfgh)$ means the determinant formed with the coordinates of p, f, g, h.) Hence, if V be the volume of the whole solid, we have

$$\Sigma \frac{(pfgh)}{pX.fX.gX.hX} = V,$$

the summation being extended over all the faces, and the summits of each so mentioned that every edge occurs twice in two different orders; that is to say, if we have mentioned $(pfgh)$, we must not mention $(pfgk)$, but $(pgfk)$ or $(pfkg)$ or $(pkgf)$. To render this equation homogeneous in all the quantities mentioned, I call to mind that the volume of a tetrahedron is not given absolutely by the formula $\frac{(pfgh)}{pX.fX.gX.hX}$, but only to a factor *près*, depending on the unit of volume employed. If we take as this unit of volume the volume of the fundamental tetrahedron, whose vertices may be denoted by 1, 2, 3, 4, then our equation becomes

$$\Sigma \frac{(pfgh)}{pX.fX.gX.hX} = V.\frac{(1234)}{1X.2X.3X.4X} \quad \ldots\ldots\ldots (1).$$

Here V is a ratio, depending on the positions of the points

$a, b, c, \ldots$ relatively to the plane X, but absolutely independent of the position of p. If, then, we make the equation integral, by multiplying both sides into $pX.1X.2X.3X.4X.\Pi.fX$, we see that the expression

$$\Sigma\,(pfgh)\frac{\Pi.fX}{fX.gX.hX}$$

must be divisible by pX; because its equivalent on the other side is so divisible, and the equation is an identity so far as p is concerned. The result of the division is of the order $n-4$ in X; or, which is the same thing, *if* X *be regarded as a variable plane, the equation*

$$\Sigma\,(pfgh)\frac{\Pi.fX}{pX.fX.gX.hX}=0\ldots\ldots\ldots\ldots\ldots(2)$$

represents a surface of class n − 4.

Two things are now clear from our previous equation and from the form of this one.

1°. If the equation is satisfied when X is the plane at infinity, then $V=0$; or, *if the surface* (2) *touch the plane at infinity, the volume of the solid is zero.*

2°. The equation (2) is satisfied if $lX=0$, $mX=0$, $nX=0$, where l, m, n are any three vertices not in the same face. Therefore *the surface* (2) *touches all the diagonal planes.*

The investigation, so far, is a mere reproduction of that of Mr Cotterill, with the addition of an extra letter to apply it to three dimensions instead of two. I shall take the liberty of calling the surface thus arrived at the *index-surface* of the polyacron, and shall denote it by the symbol v_{n-4}.

The index-surface contains all the diagonal lines. For let ab be a diagonal line, and c any other summit of the solid; then abc is a diagonal plane. For if it were a face, ab would be an edge, contrary to the supposition. Consequently $n-2$ diagonal planes can be drawn through every diagonal line; now all these are touched by the index-surface. But if more than $n-4$ tangent planes can be drawn through a straight line to a surface of class $n-4$, the line must lie altogether in the surface.

Through an edge of the solid, on the other hand, two faces and $n-4$ diagonal planes can be drawn; which latter are, of course, the tangent planes from it to the surface.

There are certain diagonal planes which it is convenient to consider separately. They are those which contain three edges of the solid; and I shall call them *single planes*. A diagonal plane may, of course, contain three diagonal lines, or two, or one, or none; but if it contains any diagonal line, the condition of touching it is already involved in the condition of containing that line. So that the facts we know about the index-surface may be summed up in saying that it passes through all the diagonal lines and touches all the single planes.

I now go on to prove that in general these conditions are precisely sufficient to determine a surface of class $n-4$. In order to do this, it will be necessary to make use of the researches of Prof. Cayley upon the Δ-faced polyacra, contained in the 1st volume of the 3rd series of the *Manchester Memoirs*, p. 248; particularly of the following passage:—

"An n-acron has n summits, $3n-6$ edges, $2n-4$ faces; and it is easy to see that there are the following three cases only, viz.:

1. The polyacron has at least one tripleural summit.
2. The polyacron, having no tripleural summit, has at least one tetrapleural summit.
3. The polyacron, having no tripleural or tetrapleural summit, has at least twelve pentipleural summits.

In fact, if the polyacron has c tripleural summits, d tetrapleural summits, e pentipleural summits, and so on, then we have

$$n = c + d + e + f + g + h + \&c.,$$

$$6n - 12 = 3c + 4d + 5e + 6f + 7g + 8h + \&c.;$$

and therefore $$12 = 3c + 2d + e + 0f - g - 2h - \&c.,$$

or $$3c + 2d + e = 12 + g + 2h + \&c.;$$

whence, if $c = 0$ and $d = 0$, then $e = 12$ at least."

Upon this theorem Prof. Cayley founds a method of deriving all polyacra with $n+1$ summits from those with n summits. If we remove from a polyacron a tripleural summit, as a in the figure [fig. 19], we may derive from it a new polyacron with one summit less by regarding the diagonal plane bcd as a face of the new solid. Conversely, we may add one summit to any polyacron by crowning any one of its faces with a tripleural summit, and then regarding this face as a diagonal plane. This process is called by Prof. Cayley the First Process. In a similar manner, the skew quadrilateral $bced$ [fig. 20] formed by two adjacent faces may be crowned by a tetrapleural summit a, with the edges ab, ac, ae, ad, the faces bcd, cde becoming diagonal planes of the new solid; this is called the Second Process. Again, the skew pentagon $bcfed$ [fig. 21] formed by three adjacent faces may be crowned by a pentipleural summit a, with the edges ab, ac, af, ae, ad, the faces bcd, cde, cef becoming diagonal planes; this is called the Third Process. And it appears from the theorem quoted above, that every $(n+1)$-acron can be made out of an n-acron by one or other of these processes, according as it belongs to the first, second, or third case of the theorem.

I shall now show, then, that if the conditions of containing the diagonal lines and touching the single planes are precisely sufficient to determine the index-surface of an n-acron, then the same thing will be true for any $(n+1)$-acron derived from it by either of these processes. This will prove that the theorem is true for all Δ-faced polyacra, provided we can show that it is true for all pentacra. Now there is only one pentacron, the figure formed of two tetrahedra with a common face, $abcde$ [fig. 22]. This figure has the diagonal line ae and the single plane bcd; and the index-surface is the point v, which is precisely determined as the intersection of these.

In determining the number of conditions involved in passing through a system of lines, we must remember that every intersection of two lines diminishes the number by one, except where three or more lines are in one plane. We have only to deal with the case of three lines in one plane; the number of conditions is then reduced by two for the three intersections.

First Process.—Let D be the number of diagonal lines of the n-acron; then when we pass to the $(n+1)$-acron, the following is the increase in the number of conditions:

The D-lines are on a surface of class $n-3$ instead of $n-4$; this makes an increase	$+D$
There are $n-3$ new diagonals joining the new summit a to all the old summits except b, c, d	$+(n-2)(n-3)$
One or other of these, however, meets each of the old ones at least once; which is all that need be counted, because if any old diagonal meets two new ones a triangle is formed	$-D$
The new diagonals all meet in a point, counting as $\frac{1}{2}(n-3)(n-4)$ intersections.....	$-\frac{1}{2}(n-3)(n-4)$
There is a new single plane bcd	$+1$
The total increase is therefore	$\frac{1}{2}(n-1)(n-2)$,

which is the difference between the number of conditions required to determine a surface of class $n-3$ and the number required for class $n-4$.

Second Process.—Let D be the number of old diagonal lines less be; then we have the following increase:

The D lines on surface of higher class.....	$+D$
There are $n-4$ new diagonals.....	$+(n-2)(n-4)$
One or other of these, however, meets each of the D lines at least once; and, as before, this is all that need be counted...	$-D$
The new diagonals all meet in a point, counting as $\frac{1}{2}(n-4)(n-5)$ intersections.....	$-\frac{1}{2}(n-4)(n-5)$
The edge cd becomes a diagonal line	$+(n-2)$
If, however, we join this edge to the $n-4$ summits different from $bcde$, there is a reduction 1 in the case of each; for either the plane was a single plane, or it contained one or two diagonals.....	$-(n-4)$
The diagonal be is on surface of higher class	$+1$
Total, as before	$\frac{1}{2}(n-1)(n-2)$.

Third Process.—Let D be the number of old diagonal lines, less df, fb, be; then we have the following increase:—

The D lines on surface of higher class.....	$+D$
There are $n-5$ new diagonals	$+(n-2)(n-5)$
Intersections of these with D lines	$-D$
New diagonals meet in a point	$-\frac{1}{2}(n-5)(n-6)$
The edges cd, ce become diagonal lines	$+2(n-2)$
If we join these to the $n-5$ summits different from $bcdef$, there is a reduction 1 for each plane	$-2(n-5)$
The diagonals df, fb, be on surface of higher class	$+3$
Their intersections with cd, ce, and of these with one another	-3
Total, as before	$\frac{1}{2}(n-1)(n-2)$.

Passing now to the consideration of polygonal faces, I remark first that, by direct application of Mr Cotterill's theorem, we have an expression for the volume of the pyramid standing on a plane polygon. For let p be the vertex of the pyramid, q any point in the plane of the polygon; then we have

$$\Sigma' \frac{(pqab)}{pX \,.\, qX \,.\, aX \,.\, bX} = U \frac{(1234)}{1X \,.\, 2X \,.\, 3X \,.\, 4X},$$

in which U is the ratio of the volume of the pyramid to that of the fundamental tetrahedron, and Σ' refers to a summation going round the m sides of the polygon in order. From this it appears that the expression

$$\frac{1}{qX} \,.\, \Sigma' \{(pqab) \,.\, \Pi' \,.\, cX\}$$

(in which $\Pi' \,.\, cX$ means a product involving all the vertices of the polygon except a and b) is integral, independent of q and of the order $m-3$ in X. If we equate it to zero, we in fact obtain the equation of Mr Cotterill's curve belonging to the polygon.

Now if this polygon form a face P of a polyacron, we obtain, as before, the following expression for the volume of the solid :—

$$\Sigma' \cdot \frac{(pqab)}{pX \cdot qX \cdot aX \cdot bX} + \Sigma \cdot \frac{(pfgh)}{pX \cdot fX \cdot gX \cdot hX} = V \cdot \frac{(1234)}{1X \cdot 2X \cdot 3X \cdot 4X}.$$

Thus the equation of the index-surface may be written

$$\frac{\Pi \cdot fX}{pX \cdot qX} \cdot \Sigma' \{(pqab) \cdot \Pi' \cdot cX\} + \frac{1}{pX} \cdot \Sigma \{(pfgh)\ \Pi \cdot aX\} = 0.$$

Here $\Pi \cdot aX$ must in every case contain $m-2$ factors at least belonging to points on the plane P; or it vanishes in the order $m-2$ when the coordinates of P are substituted in it. The term Σ' vanishes as we have seen in the order $m-3$ in the same case. Thus P is a multiple tangent plane of order $m-3$, and the curve of contact is determined by the term Σ'; that is to say, it is Mr Cotterill's curve belonging to the polygon.

Fig. 18.

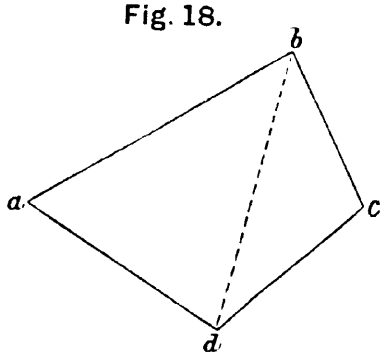


Fig. 19.

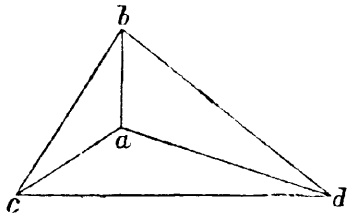


Fig. 20.

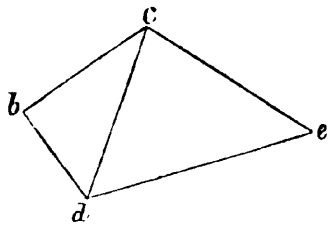


Fig. 21.

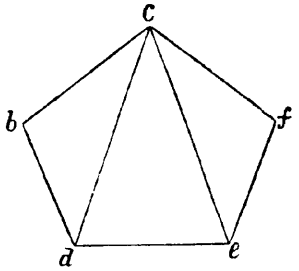
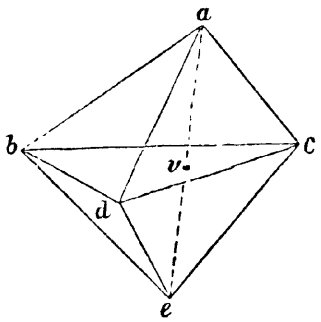


Fig. 22.



face p. 176.

XIX.

GEOMETRY ON AN ELLIPSOID*.

THE metric properties of an ellipsoid are entirely determined by the four points in which it is met by the imaginary circle at infinity. I shall start, therefore, by assuming the existence of these four coplanar points o_1, o_2, o_3, o_4, which, taken all together, I call the absolute.

1. To represent the ellipsoid on a plane we require also two fixed points i, j; the plane sections of the ellipsoid are then represented by conics through these points, and the generating lines by lines on the plane through them. In fact, if we take a fixed point a on the ellipsoid E_2, and draw a line through a and a variable point x on the ellipsoid, this line will meet a plane L in one point y, which is the representative of x; the points i, j will then represent the generators through a. If we take the points i, j to be the absolute of the plane L, then all the plane sections will be represented by circles, the lines through i will represent one system of generating lines, and the lines through j the other. We shall have then, in addition, to consider the four points o; and the geometry of the ellipsoid will be merely the geometry of the plane considered in relation to these four points, which are concyclic.

2. We know, then, that the antipoints of the o lie upon three new circles, orthotomic of each other and of the first. These correspond to the principal sections of the ellipsoid. The antipoints themselves represent umbilici, four of which are real;

* [From the *Proceedings of the London Mathematical Society*, Vol. IV. No. 54, pp. 215—217.]

I call these u_1, u_2, u_3, u_4, and we may now take the points u as our absolute instead of the points o.

3. What now are the directions of the lines of curvature at any point x of the ellipsoid? First, the indicatrix at x is represented by the point-circle at its corresponding point y, so that conjugate directions at x correspond to rectangular directions at y. Next, the tangent plane at x meets the plane at infinity in a line, say β. Through the points o can be drawn two conics to touch β, say at p, q. The lines xp, xq are tangents to the lines of curvature at x, since the points p, q are conjugates both of the section of the ellipsoid at infinity and of the imaginary circle. But now let o_1o_2 and o_3o_4 meet β in r, s respectively. Then the involution made by rs and the points where β meets the imaginary circle have p, q for double points. The interpretation of this on the plane is, that the directions through y corresponding to xp, xq make equal angles with the circles yo_1o_2, yo_3o_4. Hence—

The lines of curvature of the ellipsoid are represented by confocal anallagmatics having the u for foci.

Sections made by two conjugate planes of the ellipsoid are represented by orthotomic circles.

4. A straight line γ in space may be denoted by the two points c_1, c_2, where it meets the ellipsoid. The sections drawn through this line will be represented by the series of coaxal circles through c_1, c_2. The sections through δ, the polar of γ, will therefore be represented by the series of coaxal circles through d_1, d_2, the antipoints of c_1, c_2. Thus, a straight line being represented by a pair of points, its polar is represented by their antipoints, as is otherwise obvious.

I denote further the principal sections and the plane at infinity by $XYZU$, which notation will serve also for the circles which represent them. Now, in general, a section of E_2 passing through a fixed point of space is represented by a circle orthotomic of a fixed circle. In particular, the points $o_1\ o_2$, $o_3\ o_4$; $o_1\ o_3$, $o_4\ o_2$; $o_1\ o_4$, $o_2\ o_3$, correspond in this way to the circles XYZ. I want now to find the interpretation on the plane of

the rectangularity of the lines γ and δ. The planes joining them to the point o_1o_2, o_3o_4 are harmonic of the lines o_1o_2, o_3o_4. Hence the circles c_1c_2X, d_1d_2X are harmonic of the circles coaxal with them and passing through o_1o_2, o_3o_4 respectively. This is to be true when X and Y are interchanged: the conditions may finally be written

$$\frac{c_1c_2YZ}{d_1d_2YZ} = \frac{c_1c_2ZX}{d_1d_2ZX} = \frac{c_1c_2XY}{d_1d_2XY}.$$

If now for d_1, d_2 we may substitute c_1, c'_1 where c'_1 is indefinitely near to c_1 *in any direction*, c_1c_2 represents the normal at c_1.

5. A circle P, orthotomic of U, represents a diametral section. Let the pole of this section be called p; p is a point at infinity. We know that it is always possible to find another point q at infinity, which is conjugate to p with respect both to the ellipsoid and to the imaginary circle. We may then endeavour to find the circle Q, of which q is the pole. Further, lines λ, μ can be drawn through p, q respectively, which are at right angles, and also conjugate polars of the ellipsoid. To represent these we must find a pair of points on P which have their antipoints on Q. These circles cut orthogonally; on each of them, then, there is a singly infinite number of point-pairs representing axes of the quadric, viz., the point-pairs determined by diameters of the other circle. That is to say, any circle P, orthotomic of U, being given, there can always be found a point q^0, such that the lines through q^0 determine on P point-pairs representing axes of the quadric.

The determination of q^0 depends on the position of the projecting point a. The generators through a meet the diametral section Q in two points; the remaining generators through these intersect on the representative of q^0.

6. I now proceed to construct Q when P is given. In the first place, Q has to be orthotomic of P and U. Next, if we draw through P and Q two new circles, one of which has o_1 o_2 for harmonics, and the other o_3 o_4, these must be harmonics of

P and Q. But a circle orthotomic of U and having $o_1\ o_2$ for harmonics, must have them for inverse points, and therefore have its centre on $o_1\ o_2$. Hence the line joining the centres of P and Q is cut harmonically by the lines o_1o_2, o_3o_4. Similarly, it is cut harmonically by o_1o_3, o_4o_2, and by o_1o_4, o_2o_3. Hence the centres of P and Q are polar opposites in regard to the quadrangle $o_1\ o_2\ o_3\ o_4$. They are therefore conjugate points in regard to the circle U.

7. Intersections of the ellipsoid by spheres are represented by anallagmatics *passing through* the four points o_1, o_2, o_3, o_4. There are two systems of real circles passing through pairs of them; these represent the circular sections. Sphero-conics are represented by such of these anallagmatics as have $XYZU$ for focal circles. To find the axes of any circle P of the U system we must then draw two such anallagmatics having double contact with P; the point of contact in pairs will represent the axes of the corresponding section.

XX.

PRELIMINARY SKETCH OF BIQUATERNIONS*.

I.

THE *vectors* of Hamilton are quantities having magnitude and direction, but no particular position; the vector AB being regarded as identical with the vector CD when AB is equal and parallel to CD and in the same sense. The translation of a rigid body is an example of such a quantity; for since all particles of the body move through equal distances along parallel straight lines in the same sense, the motion is entirely specified by a straight line of the given length and direction drawn through any point whatever. A couple, again, may be adequately represented by a vector; since the axis of a couple is any line of length proportional to its moment drawn perpendicular from a given face of its plane.

For many purposes, however, it is necessary to consider quantities which have not only magnitude and direction, but *position* also. The rotational velocity of a rigid body is about a certain definite axis, and equal rotations about two parallel axes are not equivalent to one another. A force acting upon a solid has a definite line of action, and equal forces acting along parallel lines differ by a certain couple. The difference between the two kinds of quantities is clearly seen when we consider the geometric calculus which is used for the study of each. In

* [From the *Proceedings of the London Mathematical Society*, Vol. IV. Nos. 64, 65, pp. 381—395.]

studying the motions of a particle or the composition of couples, the only construction required is that of the "force-polygon," and the theory involved is that of the addition of vectors; but in the static or kinematic of solids we require in addition the construction of the "link-polygon," and there is involved the theory of the involution of lines in space, or of the linear complex.

The name *vector* may be conveniently associated with a velocity of *translation*, as the simplest type of the quantity denoted by it. In analogy with this, I propose to use the name *rotor* (short for *rotator*) to mean a quantity having magnitude, direction, and position, of which the simplest type is a velocity of *rotation* about a certain axis. A rotor will be geometrically represented by a length proportional to its magnitude measured upon its axis in a certain sense. The rotor AB will be identical with CD if they are in the same straight line, of the same length, and in the same sense; *i.e.* a vector may move anywise parallel to itself, but a rotor *only* in its own line.

The *addition* of rotors will proceed by the rules which govern the composition of forces and rotations. Here, however, we come upon a very important break in the analogy between rotors and vectors. While the sum of any number of vectors is always a vector, it will only happen in special cases that the sum of a number of rotors is a rotor. In fact, the composition of two forces whose lines of action do not intersect, or of two rotation-velocities whose axes do not intersect, gives rise to a system of forces on the one hand, and the most general velocity of a rigid body on the other. These still more complex quantities have been studied, and the theory of their addition or composition completely worked out, by Dr Ball.

A system of forces may be reduced in one way to a single force P, and a couple G whose plane is perpendicular to the line of action of the force, or *central axis*. Dr Ball speaks of the system of forces as a *wrench* about a certain *screw*; the axis of the screw being the central axis, and the pitch being the ratio $\frac{G}{P}$ of the couple to the force. Similarly the velocity of a

rigid body may be represented in one way only as a rotation-velocity ω about a certain axis combined with a translation-velocity v along that axis. Dr Ball speaks of this velocity as a *twist*-velocity about a certain screw; the axis of the screw being the axis of rotation, and its pitch the ratio $\frac{v}{\omega}$ of the translation to the rotation. A *screw* is here a geometrical form resulting from the combination of an *axis* or straight line given in position with a *pitch* which is a linear magnitude. A *wrench* is the association with this geometrical form of a magnitude whose dimensions are those of a force; a *twist*-velocity the association of a magnitude whose dimensions are those of an angular velocity. The extreme convenience of this nomenclature is well exemplified in the remarkable memoir above referred to.

Just as a vector (translation-velocity or couple) is magnitude associated with direction, and as a rotor (rotation-velocity or force) is magnitude associated with an axis; so this new quantity, which is the sum of two or more rotors (twist-velocity or wrench), is magnitude associated with a screw. Following up the analogy thus indicated, I propose to call this quantity a *motor;* the simplest type of it being the general motion of a rigid body. And we shall say that in general the sum of rotors is a motor, but that in particular cases it may degenerate into a rotor or a vector.

II.

A *quaternion* is the ratio of two vectors, or the operation necessary to make one into the other. Let the vectors be [fig. 23] AB and AC, as they may both be made to start from any arbitrary point A. Then AB is made into AC by turning it round an axis through A perpendicular to the plane BAC until its direction coincides with that of AC, and then magnifying or diminishing it until it is of the same length as AC. The ratio of two vectors then is the combination of an ordinary numerical ratio with a *rotation;* or, as Hamilton expresses it, a quaternion is the product of a tensor and a versor. Since the point A is perfectly arbitrary, this rotation is not about a definite axis;

but is completely specified when its angular magnitude and the direction of its axis are given.

This quaternion* $\frac{AC}{AB} = q$, then, is an operation which, being performed on AB, converts it into AC, so that $q \,.\, AB = AC$. The axis of the quaternion is perpendicular to the plane BAC; and it is clear that the quaternion operating upon any other vector AD in this plane will convert it into a fourth vector AE in the same plane, the angle DAE being equal to BAC and the lengths of the four lines proportionals. But a quaternion can *only* operate upon a vector which is perpendicular to its axis. If AF be any vector not in the plane BAC, the expression $q \,.\, AF$ is absolutely unmeaning. A meaning is indeed subsequently given to an analogous expression *in which the signification of* AF *is different.* But it is very important to remark that so long as AF means a *vector* not perpendicular to the axis of q, the expression $q \,.\, AF$ has no meaning at all.

Let us now consider what is the operation necessary to convert one *rotor* into another. There is one straight line which meets at right angles the axes of any two rotors, and part of which constitutes the shortest distance between them. Let AC [fig. 24] be the shortest distance between the rotors AB and CD. Then AB may be converted into CD by a process consisting of three steps. First, turn AB about the axis AC into the position AB', parallel to CD. Then slide it along this axis into the position CD'. Lastly, magnify or diminish it in the ratio of CD' to CD. The first two operations may be regarded as together forming a twist about a screw whose axis is AC and whose pitch is

$$\frac{AC}{\text{circ. meas. of } BAB'}.$$

The ratio of two rotors, then, is the combination of an ordinary

* Professor Cayley, by a very convenient notation, distinguishes $\frac{\lfloor AC}{AB\rfloor}$ and $\frac{AC}{\lfloor AB}$; viz., $AB\,\frac{AC}{AB\rfloor} = 1$, but $\frac{AC}{\lfloor AB}\,AB = 1$. It should, I think, be a convention that $\frac{X}{Y}$ is *always* to mean $\frac{X\rfloor}{\lfloor Y}$, viz., the operation which converts Y into X, *or* which, coming after the operation Y, is equivalent to the operation X.

numerical ratio with a *twist*. This twist is associated with a perfectly definite screw, and is only specified when its angular magnitude and the screw (involving direction, position, and pitch) are given. We may say also that just as the rotation (versor) involved in a quaternion is the ratio of two directions, so the twist involved in the ratio of two rotors is really the ratio of their axes.

Here again a remark must be made about the range of this operation. Using the expression *tensor-twist* to mean the ratio of two rotors (which is in fact a twist multiplied by a tensor), we may say that a tensor-twist can operate upon any rotor which meets its axis at right angles. Let t denote the operation which converts AB into CD, so that $t = \frac{CD}{AB}$, and $t \,.\, AB = CD$; then if EF be any other rotor which meets AC at right angles, the expression $t \,.\, EF$ will have a definite meaning, viz., it will mean a rotor obtained by sliding EF along a distance equal to AC, turning it about AC as axis through an angle equal to BAB', and altering its length in the ratio $AB : CD$. But if EF be a rotor not meeting AC, or meeting it at any other than a right angle, the expression $t \,.\, EF$ will have no meaning whatever.

We have now defined the ratio of two rotors, and shown that like a quaternion it has a restricted range of operation. The question naturally arises, What now is the operation which converts one *motor* into another? We can answer this question very easily in the case in which the two motors have the same pitch; for in this case their ratio is a tensor-twist whose tensor is the ratio of their magnitudes and whose twist is the ratio of their axes. We are led to this by considering each motor as the sum of two rotors which do not intersect. Let α and β be two such rotors, t a tensor-twist whose axis meets them both at right angles; then $t\alpha$ is a rotor, say γ, and $t\beta$ is another rotor, say δ. If therefore we assume the distributive law, we have

$$t\,(m\alpha + n\beta) = m\gamma + n\delta,$$

or

$$t = \frac{m\gamma + n\delta}{m\alpha + n\beta}.$$

It is a mere translation of known theorems to say that the axis of t meets at right angles the axes of the motors $m\alpha + n\beta$ and $m\gamma + n\delta$, and that one of these axes is converted into the other by the same twist that makes α into γ or β into δ.

The solution of this problem in the general case in which the pitches are different, is not so easy. In the first place, we must remember that every motor consists of a rotor part and a vector part, and that its pitch is determined by the ratio of these two parts. By combining a suitable vector with a motor, therefore, we may make the pitch of it anything we like, without altering the rotor part. Now let it be required to find the operation which will convert a motor A into a motor B. Let B' be a motor having the same rotor part as B, and the same pitch as A; and let $B = B' + \beta$, where β is a vector parallel to the axis of B. Then the ratio $\frac{B}{A} = \frac{B'}{A} + \frac{\beta}{A}$; but $\frac{B'}{A}$ is a tensor-twist, say t, and we may write

$$\frac{B}{A} = t + \frac{\beta}{A},$$

where it now only remains to find an operation which will convert a motor A into a vector β.

In order to do this, we must introduce a symbol whose nature and operation will at first sight appear completely arbitrary, but will be justified in the sequel. *The symbol ω, applied to any motor, changes it into a vector parallel to its axis and proportional to the rotor part of it.* That is to say, it changes rotation about any axis into translation parallel to that axis, and a force into a couple in a plane perpendicular to its line of action. But if the rotation is accompanied by translation or the force by a couple, the symbol takes no account whatever of these accompaniments; and if made to operate directly on a vector, reduces it to zero. It follows from this that if it be made to operate twice upon a motor, it reduces it to zero; or $\omega^2 A = 0$ always. The portion of any expression which involves ω must therefore be treated as an infinitesimal of the first order; all higher orders being uniformly neglected.

Since then $\omega A = \alpha$, a vector, and the ratio $\frac{\beta}{\alpha}$ is a quaternion q so that $q\alpha = \beta$, we may write successively

$$\beta = q\alpha = q\omega A,$$

$$\frac{\beta}{A} = q\omega,$$

and then

$$\frac{B}{A} = t + q\omega,$$

or *the ratio of two motors may be expressed as the sum of two parts, one of which is a tensor-twist, and the other is ω multiplied by a quaternion.*

The same ratio may be expressed in another form. Let an arbitrary point O be assumed as the origin; then every motor may be expressed in one way as the sum of a rotor passing through O and a vector. Now the theory of rotors passing through a fixed point is exactly the same as that of vectors in general, and the ratio of any two of them is a tensor-twist whose pitch is zero, or what is the same thing, a quaternion whose axis is constrained to pass through the fixed point. If we use cursive Greek letters (as α, β) in general to represent rotors through the origin, we may distinguish vectors from them by prefixing the symbol ω; thus $\omega\alpha$ denotes a vector parallel and proportional to the rotor α. The ratio $\frac{\beta}{\alpha}$ will then be a quaternion q, which is also the ratio $\frac{\omega\beta}{\omega\alpha}$ *. The general expression for a motor is then $\alpha + \omega\beta$. Let it now be required to find the ratio of two motors $\alpha + \omega\beta$, $\gamma + \omega\delta$; or the value of the expression

$$\frac{\gamma + \omega\delta}{\alpha + \omega\beta}.$$

First, let $\frac{\gamma}{\alpha} = q$; then $q(\alpha + \omega\beta) = \gamma + q\omega\beta = \gamma + \omega q\beta$.

The symbol $q\beta$ has at present no geometrical meaning; for in general the rotors α, β, γ will not be coplanar, and cannot

* It follows from this that $\omega q = q\omega$, or ω is commutative with quaternions.

therefore be operated on by the same quaternion q. If however (as in the Calculus of Quaternions) we consider all these quantities as expressed in terms of three rectangular unit rotors through the origin, $\frac{\delta - q\beta}{\alpha}$ will be a perfectly definite quaternion r. The equation

$$r\alpha = \delta - q\beta$$

is, like the equation $q(\alpha + \omega\beta) = \gamma + \omega q\beta$,

at present purely literal and devoid of meaning. Yet if (remembering the properties of the symbol ω) we add ω times the first equation to the second and assume the distributive law, we obtain

$$(q + \omega r)(\alpha + \omega\beta) = \gamma + \omega\delta.$$

In this way the ratio $\frac{\gamma + \omega\delta}{\alpha + \omega\beta}$ is expressed in the form $q + \omega r$, which expression may conveniently be called a *biquaternion**. The final equation, however, is not susceptible of interpretation in the same sense as the equation $q\alpha = \gamma$. The expression $q + \omega r$ does not denote the sum of geometrical operations which can be applied to the motor $\alpha + \omega\beta$ as a whole; and the ratio of two motors is only expressed by a symbol as the sum of two parts, each of which separately has a definite meaning in certain other cases, but not in the case in point. In following sections this difficulty will be partly overcome by showing that the system here sketched is the limit of another in which it does not occur.

The preceding remarks may however explain, and be illustrated by, the following table:—

Geometrical Form	Quantity	Example	Ratio
Sense on st. line	Vector on st. line	Addition or Subtraction	Signed Ratio
Direction in plane	Vector in plane	Complex quantity	Complex Ratio
Direction in space	Vector in space	Translation, Couple	Quaternion
Axis	Rotor	Rotation-Velocity, Force	Twist
Screw	Motor	Twist-Velocity, System of Forces	Biquaternion

* Hamilton's *biquaternion* is a quaternion with complex coefficients; but it is convenient (as Prof. Peirce remarks) to suppose from the beginning that all

III.

That geometry of three-dimensional space which assumes the Euclidian postulates has been called by Dr Klein the *parabolic* geometry of space, to distinguish it from two other varieties, which assume uniform positive and negative curvature respectively, and which he calls the *elliptic* and *hyperbolic* geometry of space. The investigations which follow involve the postulates of elliptic geometry. As, however, the postulate of uniform positive curvature is not sufficient to define this, it may be worth while to devote a short space to an explanation of its nature.

Space of three dimensions is that the points of which may be associated with systems of values of three variables x, y, z. It is not in general possible, however, so to make this association that to every system of values there shall correspond in general one point, and to every point in general one system of values. When this is the case, the space is called *unicursal*. An *algebraic* space is one in which the position of a point may be uniquely defined by a set of values of periodic algebraic integrals, without exceptions which form a part of the space. Thus, unicursal spaces are a particular case of algebraic. Attending now to unicursal spaces only, we must observe that there are in general exceptions to the unique correspondence of points and value-systems; namely, there are certain points to each of which correspond an infinite number of values of the coordinates satisfying a certain equation or equations; and there are certain value-systems to which correspond, not points, but loci in the space. The assignment of these point-equations and loci-values and of their relations with one another serves to determine the *projective-connection* of the space; and when once these are known, the whole of its projective geometry may be worked out. The point-equations and loci-values may or may not involve imaginary values of the variables or their coefficients; but in all cases they must be taken into account. The

scalars may be complex. As the word is thus no longer wanted in its old meaning, I have made bold to use it in a new one.

points which correspond to real systems of values are called real points; those which correspond to imaginary systems, imaginary points: the study of these latter, which does not strictly belong to that of three-dimensional space, is undertaken only for the sake of the former.

Loci which correspond to linear equations between the coordinates may at present be called *planes*, and their intersections *lines;* this is a purely projective definition, and these loci are not necessarily *flat* planes and *straight* lines in the metrical sense. Points, lines, and planes are included in the name *elements.*

The *metric* geometry of space* is the theory of the projective relations of certain fixed geometrical forms with all other geometrical forms, or of the invariant relations of certain fixed algebraic forms with all other algebraic forms. The word *power* will be explained as much as is wanted in the sequel; meanwhile it may be said that these fixed forms (called all together *the absolute*) are given when we know the points, the lines, and the planes of the absolute, or say the elements of the absolute; and that the power of an element of the absolute in regard to any arbitrary element is infinite. In other words, we *require* in general equations of the absolute in point-, line-, and plane-coordinates respectively.

A unicursal space the points of which may be represented uniquely by value-systems of the coordinates x, y, z, without the exception of any point-equations or loci-values, is called a *linear* space. This is merely a projective definition, and leaves the absolute, therefore the whole of metric geometry, undetermined.

There is a particular determination of the absolute in a linear space which is of the utmost importance. It is that in which the points of the absolute are those of a certain quadric surface, while the lines and planes of the absolute are those which touch this surface; or in which the three equations of

* This theory of metric geometry is due to Prof. Cayley: "Sixth Memoir on Quantics," *Phil. Trans.*, 1859.

the absolute are of the second degree. There are three cases* to be considered, as being the only ones of which observed space can form a part:—

(1) *Elliptic* geometry; all the elements of the absolute are imaginary.

(2) *Hyperbolic* geometry; the absolute contains no real straight lines, and surrounds us. In this case, real points situate on the other side of the surface are called *ideal.*

(3) *Parabolic* geometry; the surface degenerates into an imaginary conic in a real plane. The points of the absolute are points in the (real) plane of this conic; the lines and planes are the imaginary lines and planes which meet and touch the conic respectively.

The *first* of these suppositions will be made in what follows. It may be well here to set down in what it consists.

(1) The space to be considered is such that there is one point of it for every set of values of the coordinates x, y, z, and one set of values for every point, without any exception whatever.

(2) There is a certain quadric surface, called the absolute, all whose points and tangent planes are imaginary. If the line joining two points a, b meet the absolute in i, j, the quantity

$$\frac{ab \,.\, ij}{\sqrt{(ai \,.\, aj \,.\, bi \,.\, bj)}} \equiv \overline{ab},$$

(which is a function of anharmonic ratios, and therefore an invariant,) is called the *power* of the points a, b in regard to one another, or of either in regard to the other. The *distance* of these two points is an angle θ such that

$$\sin \theta = \overline{ab}.$$

Similarly, if through the line of intersection of the planes A, B there be drawn the tangent planes I, J to the absolute,

* On this division see Dr Klein, "Ueber die so-genannte Nicht-Euklidische Geometrie," *Math. Annalen*, Bd. 4. The second case is the geometry of Lobatschewsky and Bolyai.

the power of the planes A, B in regard to one another is the quantity

$$\frac{AB \,.\, IJ}{\sqrt{(AI \,.\, AJ \,.\, BI \,.\, BJ)}} = \overline{AB},$$

and the angle between them is an angle ϕ such that

$$\sin \phi = \overline{AB}.$$

(3) If two points are conjugate in regard to the absolute, they are distant a *quadrant* from one another; if two lines or planes are conjugate in regard to the absolute, they are at right angles. Thus all the points at a quadrant distance from a given point are situate on its polar plane in regard to the absolute, and every plane through it cuts this polar plane at right angles. Every line has a polar line in regard to the absolute, such that every point on the polar line is distant a quadrant from every point on the line; and every line which is at right angles to either meets the other. Through an arbitrary point can in general be drawn *one* line perpendicular to a given plane; namely, the line joining the point to the pole of the plane. If, however, the point *is* the pole of the plane, every line through it is perpendicular to the plane. Similarly, from a point not on the polar of a given line can be drawn one and only one perpendicular to the line; namely, the line through the point which meets the given line and its polar.

(4) *In general, two lines can be drawn so that each meets two given lines at right angles, and these are polars of one another.* One line may therefore be converted into another by rotation about two polar axes. These axes are determined as the lines which meet the two given lines and their polars. If we travel continuously along one of these lines and draw perpendiculars on the other, one of these axes determines the shortest distance between the lines, and the other the longest. If then these two are equal, the lines are equidistant along their whole length. Thus *there is a case of exception in which two lines and their polars belong to the same set of generators of a hyperboloid; the lines are then equidistant along their whole length, and meet the same two generators of one system of the*

absolute. I shall use the word *parallel* to denote two lines so situated; and they shall be called *right* parallel or *left* parallel according as one is converted into the other by a right-handed or left-handed twist. Through an arbitrary point can be drawn one right parallel and one left parallel to a given line; the angle between them is twice the distance of the point from the line. There are many points of analogy between the *parallels* here defined and those of parabolic geometry. Thus, if a line meet two parallel lines, it makes equal angles with them; and a series of parallel lines meeting a given line constitute a ruled surface of zero curvature. The geometry of this surface is the same as that of a finite parallelogram whose opposite sides are regarded as identical.

(5) A twist-velocity of a rigid body must be regarded as having *two* axes. For a motion of translation along any axis is the same thing as a rotation about the polar axis, and *vice versâ.* Hence a twist-velocity is compounded of rotation-velocities about two polar axes; say these are θ, ϕ. Then the motion may be regarded either as a twist-velocity about a screw whose pitch is $\frac{\phi}{\theta}$ and whose axis is the first axis, or about a screw whose pitch is $\frac{\theta}{\phi}$ and whose axis is the polar axis. In general, then, a motor has two axes, and is expressible in one way only as the sum of two polar rotors. There is, however, one case of exception in which the axes of a motor are indeterminate; that, namely, in which the magnitudes of the two polar rotors are equal*. If a rigid body receive at the same time a rotation about an axis and an equal translation along it, all the points of the body will describe parallel straight lines; and the motion of the body is at the same time a rotation about any one of these lines combined with an equal translation along it. Such a motion may be adequately represented by a line of given length drawn through any point whatever parallel to a given line. A motor of pitch unity, or which is its own polar, may therefore

* This motion is described in another connection by Drs Klein and Lie, *Math. Annalen*, Bd. 4; it is a transformation of the absolute into itself in which two generators remain unaltered.

be regarded as having the nature of a *vector*, and shall in future be denoted by that name. For we may define a vector as a motor whose axes are indeterminate; and the case we are now considering is the only case of such indetermination which occurs in elliptic geometry. Vectors will be called *right* or *left* according as the twist of them is right- or left-handed.

Prop.: *Every motor is the sum of a right and a left vector.* For let A be a motor, and A' the polar motor; then we have $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$. Now $A + A'$ and $A - A'$ are both motors of pitch unity, but one right-handed and the other left-handed.

IV.

A fixed point being chosen as origin, let three lines perpendicular to one another be drawn through it, and let three unit-rotors having these lines as axes be denoted by the symbols i, j, k. Then every rotor through the origin will be denoted by an expression of the form $ix + jy + kz$, where x, y, z are scalar quantities, or the ratios of magnitudes. The symbols i, j, k shall have also another meaning; viz., each shall signify the rotation through a right angle about its axis of any rotor which meets that axis at right angles. When they are performed on rotors passing through the origin, these operations satisfy the equations $i^2 = j^2 = k^2 = ijk = -1$, by the ordinary rules of quaternions; and it is easy to see that the same equations hold good when the operations are performed on rotors not passing through the origin. The compound symbol $ix + jy + kz$ is also to have an analogous secondary meaning; viz., a rectangular rotation about the axis of the rotor which it previously denoted, combined with a tensor $\sqrt{(x^2 + y^2 + z^2)}$. It can operate only on a rotor which meets its axis at right angles. This being so, the ratio of any two rotors through the origin is a *quaternion* of the form $q \equiv w + ix + jy + kz \equiv w + \rho$, say. The axis ρ of this quaternion is perpendicular to the plane of the two rotors. If α be a rotor through the origin and q a quaternion, the product $q\alpha$ can be formed according to the Hamiltonian rules of multiplication, and is in general a quaternion r. In this general case

the equation $q\alpha = r$ can only be interpreted by giving to α its *secondary* meaning; and the translation of this statement into words is as follows:—If a rotor be capable of being successively operated upon by the rectangular versor α and the quaternion q, the final result will be the same as if it had been originally operated upon by the quaternion r. If, however, the axes of q and α are at right angles, the scalar part of r will be wanting, and we may write the equation $q\alpha = \rho$. This equation is now susceptible of a *primary* interpretation; viz., the quaternion q operating on the rotor α produces the rotor ρ; although the *secondary* interpretation does not cease to be true.

With such conventions, the two sides of the equation

$$(q + r)\, s = qs + rs$$

(in which q, r, s are quaternions) have always the same meaning when both are interpretable; which is what is meant by saying that the distributive law holds good for these symbols.

The ratio of two rotors which do not meet is a twist which in general has perfectly definite axes. But when the rotors are polars of one another, the axes of the twist are indeterminate; for any line meeting both meets them at right angles, and will serve for an axis. It is therefore always possible to find a twist which shall simultaneously convert two given rotors into their polars; and any two rectangular twists with pitch 1 or -1 have a pair of common rotors on which they can operate, and which they convert into one another. Hence we may consider that

All rectangular twists of pitch 1 *are equivalent to one another; and all rectangular twists of pitch* -1 *are equivalent to one another.*

The rectangular twist of pitch 1 shall be denoted by the symbol ω; the expression $\omega\alpha$ will denote the rotor polar to α and equal to it in magnitude, obtained from it by a left-handed twist. During the operation of this twist, every point of the rotor describes a straight line; if therefore the twist be continued through two right angles, the rotor will be replaced in its original position, *not* reversed; we have therefore

$$\omega^2 = 1.$$

Every motor can be expressed as the sum of two rotors, one passing through the origin and the other being polar to a rotor through the origin. The general expression for a motor is therefore $\alpha + \omega\beta$.

This will represent a *rotor* if the two rotor constituents intersect, or if each is perpendicular to the polar of the other; or if $S\alpha\beta = 0$.

Let now $$\xi = \frac{1+\omega}{2}, \quad \eta = \frac{1-\omega}{2};$$

then $$\xi^2 = \frac{1+2\omega+\omega^2}{4} = \frac{2+2\omega}{4} = \xi,$$

$$\eta^2 = \frac{1-2\omega+\omega^2}{4} = \frac{2-2\omega}{4} = \eta,$$

$$\xi\eta = \frac{1-\omega^2}{4} = 0.$$

Any motor $\alpha + \omega\beta$ can also be expressed in the form $\xi\gamma + \eta\delta$. It is clear that $\xi\gamma$ is the right vector part of this motor, and that $\eta\delta$ is the left vector part. If we multiply $\xi\gamma + \eta\delta$ by ξ, the result is merely $\xi\gamma$; so the effect of multiplying a motor by ξ is merely to pick out the right vector part of it. The symbols ξ, η are thus in a certain sense *selective* symbols, and are analogous to the S and V of quaternions.

Ratio of two motors.—We can find immediately now the operation which converts a motor $\xi\gamma + \eta\delta$ into a motor $\xi\alpha + \eta\beta$. For if we perform the operation

$$\left(\xi\frac{\alpha}{\gamma} + \eta\frac{\beta}{\delta}\right)(\xi\gamma + \eta\delta),$$

remembering the laws of multiplication of ξ, η, we obtain the result $\xi\alpha + \eta\beta$. If then $\frac{\alpha}{\gamma} = q$, $\frac{\beta}{\delta} = r$, we may write

$$\frac{\xi\alpha + \eta\beta}{\xi\gamma + \eta\delta} = \xi\frac{\alpha}{\gamma} + \eta\frac{\beta}{\delta} = \xi q + \eta r,$$

and the latter may be written in the form

$$\frac{q+r}{2} + \omega\cdot\frac{q-r}{2} = s + \omega t,$$

showing that *the ratio of two motors is a biquaternion.*

The motor $\xi\alpha + \eta\beta$ will be a *rotor* if

$$S(\alpha+\beta)(\alpha-\beta) = 0,$$

or if

$$T\alpha = T\beta;$$

and it is easy to see from this that the biquaternion $\xi q + \eta r$ will be a *twist*, or the ratio of two rotors, if $Tq = Tr$.

V.

1. *Position-Rotor of a Point.*—The coordinates of a point in regard to a quadrantal tetrahedron 1234 being x_1, x_2, x_3, x_4, the equation to the absolute is $\Sigma x^2 = 0$. The rotor from the origin (the point 4) to the point x is represented by

$$i_1\frac{x_1}{x_4} + i_2\frac{x_2}{x_4} + i_3\frac{x_3}{x_4}, \text{ or } \Sigma i_k\frac{x_k}{x_4}\,(k = 1, 2, 3),$$

where i_1, i_2, i_3 are rotors along the edges of the tetrahedron from the origin to the middle points of the edges. The tensor of this rotor is the tangent of the angular distance from the origin to the point it represents. For if

$$\rho = i_1\frac{x_1}{x_4} + i_2\frac{x_2}{x_4} + i_3\frac{x_3}{x_4},$$

$$(T\rho)^2 = \frac{x_1^2 + x_2^2 + x_3^2}{x_4^2} = \tan^2\widehat{ox}, \text{ where } o \text{ is the origin.}$$

The angular distance from the origin to a point has an infinite number of values, which differ by multiples of π. If therefore a rotor be considered to have this angular distance as its length, the rotor of a point can only be defined by such an equation as $\breve{\rho} \equiv \breve{\alpha} \pmod{\breve{\pi}_a}$. To obviate this indetermination, there is required a one-valued unicursal function having the period π; the tangent of the angular distance is hereby completely singled out.

2. *Equation of a Straight Line.*—Let OM [fig. 25] be the perpendicular from the origin O upon the straight line MP;

and let ON be a line perpendicular to OM in the plane MOP. Then from the triangle MOP we have

$$\frac{\tan OM}{\tan OP} = \cos MOP;$$

or if $\quad OM = \alpha,\ OP = \rho,\ ON = \beta,\ T\alpha = T\rho \cos MOP;$

so that α is the component of ρ in the direction OM, and we have $\rho = \alpha + \beta x$, where x is some scalar.

By varying x, then, we get all the points in the line MP. But if α_1 is *any* particular value of ρ, the equation may just as well be written $\quad \rho = \alpha_1 + \beta x,$

where now α_1 is not necessarily perpendicular to β.

This form may be reduced to the preceding as follows:

To find the perpendicular from O, put $\delta T\rho = 0$; this gives

$$S\alpha_1\beta + \beta^2 x = 0,$$

and the equation becomes

$$\rho = \alpha_1 - \beta S\frac{\alpha_1}{\beta} - \beta x,$$

where $\alpha_1 - \beta S\frac{\alpha_1}{\beta} = \alpha$ of the former equation.

3. *Rotor along Straight Line whose Equation is given.*

Let OR [fig. 26] be the rotor through the origin which has right parallelism with MP. Then $\angle NOR = OM$. Let OK be perpendicular to ON and OM, and of such length that

$$\frac{\tan OK}{\tan ON} = \tan NOR.$$

Then, if $\quad \gamma = OK,\ OR = \beta + \gamma.$

Now $\frac{T\gamma}{T\beta} = T\alpha$, and $U\gamma = U\alpha\beta$, since γ is perpendicular to α and β. Hence $\gamma = \alpha\beta$; and if R be a rotor along MP, m a scalar,

$$\text{right vector of } R = \xi R = m\xi(\beta + \gamma) = m\xi(\beta + \alpha\beta),$$

so $\quad$ left vector of $R = \eta R = m\eta(\beta - \gamma) = m\eta(\beta - \alpha\beta)$;

therefore $\quad R = m(\beta + \omega\alpha\beta).$

Now if R have the same length as β, we have

$$\beta^2 = R^2 = m^2(\beta^2 + \overline{\alpha\beta}^2) = m^2\beta^2(1-\alpha^2);$$

therefore
$$R = \frac{\beta + \omega\alpha\beta}{\sqrt{(1-\alpha^2)}}.$$

Conversely, equation to axis of rotor $\gamma + \omega\delta$ is

$$\rho = \frac{\delta}{\gamma} + \gamma x.$$

This finds the rotor in the case in which $\rho = \alpha + \beta x$, where $S\alpha\beta = 0$. But in the general case we have only to write the equation in the form

$$\rho = \alpha - \beta S\frac{\alpha}{\beta} + \beta x,$$

whence
$$R = \frac{\beta + \omega\left(\alpha - \beta S\frac{\alpha}{\beta}\right)\beta}{\sqrt{\left(1-\alpha^2-\beta^2S^2\frac{\alpha}{\beta} + 2S\alpha\beta S\frac{\alpha}{\beta}\right)}}$$

$$= \frac{\beta + \omega V\alpha\beta}{\sqrt{\left(1 + S\alpha\beta S\frac{\alpha}{\beta} - \alpha^2\right)}}.$$

4. *Rotor* ab *joining Points whose Position-Rotors are* α, β.

The equation of this rotor is

$$\rho = \alpha + (\beta - \alpha)x,$$

whence
$$mR = \beta - \alpha + \omega V\alpha\beta.$$

Now if a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4 are the coordinates of the points, we have

$$(TR)^2 = \tan^2 ab = \frac{\Sigma(a_hb_k - a_kb_h)^2}{(\Sigma a_hb_h)^2} = -\frac{(\alpha-\beta)^2 + (V\alpha\beta)^2}{(1 - S\alpha\beta)^2},$$

therefore
$$R = \frac{\beta - \alpha + \omega V\alpha\beta}{1 - S\alpha\beta}.$$

Cor.—If ρ [fig. 27] be the rotor of a variable point on a curve, $d\lambda$ a rotor along the tangent of length equal to the arc of the curve between ρ and $\rho + d\rho$, we have

$$d\lambda = \frac{d\rho + \omega V\rho d\rho}{\sqrt{1-\rho^2}}.$$

5. *Rotor parallel to β through Point whose Position-Rotor is α.*

The general equation to a line through the point α is $\rho = \alpha + \lambda x$, where λ is any rotor through the origin. A rotor along this line is $\lambda + \omega V\alpha\lambda$; if this is right parallel to β, we have

$$\xi(\lambda + V\alpha\lambda) = \xi\beta, \qquad (\xi\omega = \xi)$$

or
$$\lambda + V\alpha\lambda = \beta.$$

Operating by $S\alpha$, we have, since $S.\alpha V\alpha\lambda = 0$,

$$S\alpha\lambda = S\alpha\beta,$$

whence, by addition, $\lambda + \alpha\lambda = \beta + S\alpha\beta$,

and $\lambda = (1+\alpha)^{-1}(\beta + S\alpha\beta) = \beta - (1+\alpha)^{-1}V\alpha\beta.$

The rotor required is

$$\lambda + \omega V\alpha\lambda, \text{ or } \lambda + \omega(\beta - \lambda).$$

This becomes, then,

$$\beta - (1+\alpha)^{-1}V\alpha\beta + \omega(1+\alpha)^{-1}V\alpha\beta = \beta - 2\eta(1+\alpha)^{-1}V\alpha\beta.$$

Instead of operating by $S\alpha$ on the equation

$$\lambda + V\alpha\lambda = \beta,$$

we might have operated with $V\alpha$, and got

$$V\alpha\lambda + \alpha V\alpha\lambda = V\alpha\beta, \text{ since } V.\alpha V\alpha\lambda = \alpha V\alpha\lambda,$$

therefore $V\alpha\lambda = (1+\alpha)^{-1}V\alpha\beta,$

and $\lambda = \beta - V\alpha\lambda = \beta - (1+\alpha)^{-1}V\alpha\beta.$

Similarly, we have for the rotor *left* parallel to β,

$$\lambda = \beta + (1-\alpha)^{-1}V\alpha\beta,$$

and the rotor is

$$\lambda + \omega(\lambda - \beta) = \beta + (1-\alpha)^{-1}V\alpha\beta + \omega(1-\alpha)^{-1}V\alpha\beta$$
$$= \beta + 2\xi(1-\alpha)^{-1}V\alpha\beta.$$

XXI.

GRAPHIC REPRESENTATION OF THE HARMONIC COMPONENTS OF A PERIODIC MOTION*.

FOURIER'S theorem asserts that any motion having the period P may be decomposed into simple harmonic motions having periods P, $\frac{1}{2}P$, $\frac{1}{3}P$, &c.; and assigns the amplitudes and phases of these motions by means of definite integrals. In fact, if $\phi(x+2\pi)=\phi(x)$ for all values of x,

then $$\phi(x)=\tfrac{1}{2}b_0+b_1\cos x+b_2\cos 2x+\ldots+b_m\cos mx+\ldots$$
$$+a_1\sin x+a_2\sin 2x+\ldots+a_m\sin mx+\ldots,$$

where $$\pi b_m=\int_{-\pi}^{+\pi}\phi(\alpha)\cos m\alpha\,d\alpha,$$

$$\pi a_m=\int_{-\pi}^{+\pi}\phi(\alpha)\sin m\alpha\,d\alpha;$$

and this is made applicable to the general case of periodic motion by putting $\frac{x}{2\pi}=\frac{t}{P}$, where t is the time elapsed since the era of reckoning.

The terms $b_m\cos mx+a_m\sin mx$ constitute a simple harmonic motion of period $\frac{P}{m}$; the object of the present communication is to represent this motion by a graphical construction.

If a right circular cylinder be made to revolve uniformly about its axis, while a pencil in contact with its surface has

* [From the *Proceedings of the London Mathematical Society*, Vol. v. No. 67, pp. 11—14.]

a rectilinear motion parallel to the axis, the pencil will trace out upon the cylinder a curve representing its motion. In particular, if this motion is simple harmonic and of a period equal to that of the revolution of the cylinder, the curve traced out will be an ellipse. The amplitude of the motion will be $r \cot \theta$, where r is the radius of the cylinder and θ the inclination of its axis to the plane of the ellipse; the phase at epoch is determined by the orientation of the major axis. This ellipse may thus be regarded as a graphical representation of the simple harmonic motion.

Now let the pencil have any arbitrary motion whose period is P. For convenience let us suppose that the axis of the cylinder is vertical. Then, if the cylinder be made to turn once round in the time P, a curve C_1 will be traced on it, representing the arbitrary periodic motion. Next let the cylinder turn round twice in the period P; a curve C_2 will be traced on it. And generally when the cylinder turns round m times in the time P, a curve C_m will be traced on it, going m times round the cylinder. All these curves C will be closed curves, because the motion is periodic.

At this point I call to mind Mr Hayward's extension of the meaning of "area," whereby it is made to have direction as well as magnitude*. Any closed contour, not necessarily plane, being given, the area of its projection on a plane is found to be a maximum when the plane has a certain aspect. The magnitude of this maximum area, considered as having this particular aspect, is called the area of the contour; and the area of the projection on any other plane is proportional to the cosine of the angle which it makes with the maximum plane. If, therefore, we know the area of the projection of a contour on any three planes at right angles, we can find the area of the projection on any other plane.

Now I say that it is possible to draw on the cylinder an ellipse which shall have the same area in magnitude and direction as the curve C_1. For the projection of this curve on a

* [*Proceedings of the London Mathematical Society*, Vol. IV. No. 59, pp. 289—291.]

plane perpendicular to the axis (*i.e.* a horizontal plane) is merely the circular section of the cylinder, which is the same as the projection of any ellipse traced on it. If therefore we cut the cylinder by a plane parallel to the maximum plane of the contour C_1, the elliptic section E_1 will have the same area as that contour in magnitude and direction.

The contour C_2 goes twice round the cylinder; therefore its projection on a plane perpendicular to the axis is the circle described twice in the same direction, and its area is twice that of the projection of any ellipse. If therefore we cut the cylinder by a plane parallel to the maximum plane of C_2, we shall obtain an ellipse E_2 whose area is half that of the contour C_2 and parallel to it. Similarly, the contour C_m goes m times round the cylinder, and a plane parallel to its maximum plane will determine an ellipse E_m whose area is $\frac{1}{m}$th of that of C_m.

Now first let a circle be drawn on the cylinder whose height is the mean height of C_1. On this circle is the middle point of all the component oscillations.

Next, while the cylinder goes round once in the period P, let the pencil follow the ellipse E_1; it will then have a simple harmonic motion of period P, which is, in fact, the first or fundamental component. Then, while the cylinder goes round twice in the time P let the pencil follow the ellipse E_2: the resulting simple harmonic motion of period $\frac{P}{2}$ is the second component. Generally, while the cylinder goes round m times in the time P, let the pencil follow the ellipse E_m; this simple harmonic motion is the mth component.

The demonstration of this result is very simple. The values of a_m and b_m may be written as follows:

$$\pi b_m = \int_{-\pi}^{+\pi} \phi x \cos mx dx = \frac{1}{m} \int_{\alpha=-\pi}^{\alpha=\pi} \phi x d(\sin mx),$$

$$\pi a_m = \int_{-\pi}^{+\pi} \phi x \sin mx dx = -\frac{1}{m} \int_{\alpha=-\pi}^{\alpha=\pi} \phi x d(\cos mx).$$

Suppose now that ϕx is set up vertically at P [fig. 28], when $\Gamma CA = m\alpha$, then $d \cos m\alpha$ is the element of CA, and $d \sin m\alpha$ is the element of CB; so that the differentials under the integral signs are respectively elements of the areas projected on vertical planes through AA' and BB'. If in these integrals we write $b_m \cos \alpha + a_m \sin \alpha$ in place of $\phi\alpha$, we get the areas of the corresponding projections of the ellipse E_m; these are πb_m and πa_m respectively. Thus the area of C_m projected on three planes at right angles is m times that of the ellipse E_m; or the areas of the two curves have the same aspect and are in the ratio $m : 1$; which was to be proved.

XXII.

ON THE TRANSFORMATION OF ELLIPTIC FUNCTIONS*.

THE following communication is an attempt to apply Jacobi's geometrical representation of the addition-theorem in elliptic functions to the theory of their transformation. For this purpose I use the said representation in the following form.

Consider two circles, one of which is wholly within the other, but which are not concentric; as in the figure [fig. 29]. The points of the outer circle may be uniquely represented by a parameter x, such that if 0 and ∞ are the points represented by these values respectively, and $0t$ is the tangent at the former, x is proportional to the ratio of the sines of the angles which $0x$ makes with the lines $0t$ and 0∞. Let the angle $x0t = \phi$; then, if we make $x = i \tan \phi$ $(i = \sqrt{-1})$, the values 1, -1 of the parameter will belong to the circular points at infinity. Let then k^{-1}, $-k^{-1}$ be the values belonging to the imaginary points of intersection of the two circles. Through the points 0, x let tangents be drawn to the inner circle, meeting the outer circle in c, ξ; these being so chosen that, when x moves continuously to 0, ξ will move continuously to c. Then Jacobi's theorem is that, if

$$x = \mathrm{sn}\,(u, k), \quad \xi = \mathrm{sn}\,(r, k), \quad c = \mathrm{sn}\,(\gamma, k), \quad \text{then } r = u + \gamma.$$

The extension to any two conics, made by Prof. Cayley, may be put into the same form. The representation of each point

* [From the *Proceedings of the London Mathematical Society*, Vol. VII. Nos. 90, 91, pp. 29—38.]

of a conic by a parameter is determinate when we know the parameters of any three points. Now the four intersections of two conics U, V may be divided into pairs in three ways, and will so determine three involutions upon the conic U. Let one of these be chosen, and let the parameters 0 and ∞ be assigned to its united points. Then, if the value 1 be assigned to one of the intersections, -1 will belong to another of them; and the remaining two will have parameters equal in magnitude but of contrary signs. Call these $\pm k^{-1}$, and draw the tangents as before; then Jacobi's theorem remains true, except that we must write $\pm \operatorname{sn}^{-1}\xi = \operatorname{sn}^{-1}x + \operatorname{sn}^{-1}c$; the sign of $\operatorname{sn}^{-1}\xi$ depending on the reality of the common tangents.

The proof of it depends on the symmetrical (2, 2) correspondence, considered by Euler, in relation to the addition of elliptic integrals. Given that the relation between the points x and ξ is that the line $x\xi$ touches the conic V, it is clear that to one position of x correspond two positions of ξ; and to one position of ξ, two positions of x; or, the points have a (2, 2) correspondence. Hence the equation connecting the quantities x, ξ must be of the second order in each; and it must obviously be symmetrical. Let the equation be

$$(ax^2 + 2bx + c)\,\xi^2 + 2\,(bx^2 + 2b'x + c')\,\xi + (cx^2 + 2c'x + c'') = 0;$$

or, which is the same thing,

$$(a\xi^2 + 2b\xi + c)\,x^2 + 2\,(b\xi^2 + 2b'\xi + c')\,x + (c\xi^2 + 2c'\xi + c'') = 0.$$

The values of x which make the two corresponding values of ξ coincide are given by the equation

$$X = (ax^2 + 2bx + c)\,(cx^2 + 2c'x + c'') - (bx^2 + 2b'x + c')^2 = 0,$$

and similarly the values of ξ which make the two corresponding values of x coincide are given by $\Xi = 0$, where Ξ is the same function of ξ that X is of x. Now, by differentiating the original equation, we easily find $dx : \sqrt{X} = d\xi : \sqrt{\Xi}$.

The roots of the equation $X = 0$ are clearly the parameters of the points of intersection of the two conics; for these are the only points on U from which two coincident tangents can be drawn to V. If, then, the parameters of these points have

been made equal to ± 1, $\pm k^{-1}$, X must be proportional to $(1-x^2)(1-k^2x^2)$, and the differential equation becomes

$$\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}};$$

whence, since, when $x=0$, $\xi=c$,

$$\pm \operatorname{sn}^{-1} x + \operatorname{sn}^{-1} c = \operatorname{sn}^{-1} \xi,$$

which is the theorem in question.

If we change V into $U+\sigma V$, and allow σ to vary, this varying conic will always have the same intersections with U, and therefore k will be constant; but c will depend upon the value of σ. It is clear that c^2 is given uniquely when σ is given, but, when c is given, there are two values of σ. When $c=0$, the two conics must touch at 0, and therefore must coincide; thus both values of σ vanish. When $c=\infty$, the conic $U+\sigma V$ becomes a pair of lines intersecting on the line $0\,\infty$; let α and β be the values of σ which belong to these, and which are, of course, roots of the equation $\square(U+\sigma V)=0$.

Then we must have $c^2 = \dfrac{m\sigma^2}{(\sigma-\alpha)(\sigma-\beta)}$,

where m is an undetermined constant.

Suppose, now, that a polygon is inscribed in U by the following process; a tangent is drawn from x to a conic σ_1, meeting U again in x_1; then from x_1 to a conic σ_2, meeting U in x_2, and so on; finally, let x_{n-1} be joined to x. If $c_1, c_2 \ldots c_{n-1}$ be the constants belonging to these conics respectively, we shall have

$$\begin{aligned}
\pm \operatorname{sn}^{-1} x_1 &= \operatorname{sn}^{-1} x + \operatorname{sn}^{-1} c_1,\\
\pm \operatorname{sn}^{-1} x_2 &= \operatorname{sn}^{-1} x_1 + \operatorname{sn}^{-1} c_2;\\
&\text{\&c.} \qquad\quad \text{\&c.}\\
\pm \operatorname{sn}^{-1} x_{n-1} &= \operatorname{sn}^{-1} x_{n-2} + \operatorname{sn}^{-1} c_{n-1};
\end{aligned}$$

whence, by addition, with proper changes of sign,

$$\begin{aligned}
\pm \operatorname{sn}^{-1} x_{n-1} &= \operatorname{sn}^{-1} x + \operatorname{sn}^{-1} c_1 \pm \operatorname{sn}^{-1} c_2 \pm \ldots \pm \operatorname{sn}^{-1} c_{n-1}\\
&= \operatorname{sn}^{-1} x + \operatorname{sn}^{-1} c_n, \text{ suppose.}
\end{aligned}$$

From this equation it appears that the last side of the polygon will always touch the same conic of the series $U+\sigma V$, wherever the starting-point x is taken. Or, if a polygon be inscribed in U, and move so that all but one of its sides touch conics of the series $U+\sigma V$, then the remaining side will also touch a conic of the series. This theorem of Poncelet's is proved in this way by Jacobi. It is to be noticed that the signs of the quantities c depend upon which tangent is drawn from the corresponding point x; the two tangents belong in a definite way to the two tangents from 0. The final value of c_n being thus determined, one of the two conics belonging to it is singled out by the sign given to $\mathrm{sn}^{-1} x_{n-1}$. In fact, the whole series of conics $U+\sigma V$ is divided into three parts by the three line-pairs it contains. For two conics in the same part of the series the signs of $\mathrm{sn}^{-1}\xi$ are certainly the same; for conics in different parts they may be different.

And it appears that the conics mentioned in the theorem may belong to any part of the series if the signs be properly chosen in each of the equations. It is remarked by Jacobi that with these restrictions the position of x_{n-1} does not depend upon the order in which the conics are taken.

Now suppose two such polygons to be drawn with the same system of conics, x being continuously moved to ξ, and at the same time x_1 to ξ_1, &c. We shall then have the equations

$$\pm \mathrm{sn}^{-1} x_1 = \mathrm{sn}^{-1} x + \mathrm{sn}^{-1} c_1,$$
$$\pm \mathrm{sn}^{-1} \xi_1 = \mathrm{sn}^{-1} \xi + \mathrm{sn}^{-1} c_1,$$

the signs being the same in both. Consequently

$$\pm (\mathrm{sn}^{-1} x_1 - \mathrm{sn}^{-1} \xi_1) = \mathrm{sn}^{-1} x - \mathrm{sn}^{-1} \xi,$$

or the lines $x_1\xi_1$, $x\xi$ touch the same conic of the series $U+\sigma V$.

Proceeding in this way, we may show that

$$\pm (\mathrm{sn}^{-1} x_r - \mathrm{sn}^{-1} \xi_r) = \mathrm{sn}^{-1} x - \mathrm{sn}^{-1} \xi;$$

whence it appears that the lines joining corresponding vertices of the two polygons all touch the same conic of the series when the n conics touched by the sides belong to the same part; but if they belong to different parts, each joining line touches one

of two conics which harmonically divide the pairs of lines $U+\alpha V$, $U+\beta V$.

Attending now only to the first case, it will be convenient to re-state the two theorems together, as follows:—

If a polygon be inscribed in a conic U *so that all its sides but one touch conics of the series* $U+\sigma V$, *the remaining side will also touch a conic of the series.*—(Poncelet's Theorem.)

When all these conics can pass continuously into one another without breaking up into two straight lines, the lines joining corresponding vertices of two such polygons will all touch a conic of the series.

Let us now consider the particular case in which all the sides of the moving polygon touch the same conic. Here the second theorem is true without restriction; the lines joining corresponding vertices of two such polygons will always touch one conic passing through the intersections of the other two. In this case also *the vertices of the variable polygon determine upon the conic* U *an involution of the* n^{th} *order;* that is to say, if the parameters of the vertices of one polygon be determined by an equation $p_n=0$ of the nth order, and those of another polygon by an equation $q_n=0$, then the vertices of any third polygon will be determined by an equation $p_n-yq_n=0$, where y is a variable quantity, which we may call the parameter of the polygon. The relation between y, the parameter of the polygon, and x, the parameter of any one of its vertices, is $y=p_n:q_n$, where p_n, q_n are rational integral functions of the nth order in x.

Suppose, then, the relation between U and V to be such that a polygon of n sides may be inscribed in U and circumscribed to V. Let $x, x_1 \ldots x_{n-1}$ be the vertices of such a polygon; then, if $x=\mathrm{sn}\,u$, we must have $x_1=\mathrm{sn}\,(u+\gamma)$, $x_2=\mathrm{sn}\,(u+2\gamma)$, $x_3=\mathrm{sn}\,(u+3\gamma) \ldots x_{n-1}=\mathrm{sn}\,\{u+(n-1)\gamma\}$, and consequently $x=\mathrm{sn}\,(u+n\gamma)$. Therefore $n\gamma$ is a period of the elliptic function; and the number of conics of the series $U-\sigma V$, which can be inscribed in n-gons inscribed in U is equal to the number of periods whose n^{th} parts are not congruent, that is, for n a prime number it is $n+1$.

Now let another polygon be drawn having a vertex at ξ, and let η be its parameter. Then the lines $x\xi$, $x_1\xi_1$ ... &c. will all touch a conic W. Let this conic be held fixed, and the two polygons moved so that the lines joining corresponding vertices always touch W. Then to every value of y will belong two values of η, and *vice versâ*, and this (2, 2) correspondence is symmetrical. Hence *a symmetrical* (2, 2) *correspondence between individual vertices implies a symmetrical* (2, 2) *correspondence between the polygons.*

Now, the parameter of every polygon is determined when we know the parameters of three polygons. Let the parameters of the polygons which have a vertex at 0, 1, and ∞ be made equal to 0, 1, and ∞ respectively; this amounts to saying that $p_n = 0$ has a root 0, $q_n = 0$ has a root ∞, and $p_n - q_n = 0$ has a root 1. It is clear, then, from the symmetry of the figure, that y must be an odd function of x, so that $p_n + q_n = 0$ will have a root -1. This amounts to saying that p_n is x multiplied by a rational integral function of x^2, and q_n (which is really only of the order $n-1$) is another rational integral function of x^2. This being so, let $\pm\lambda^{-1}$ be the parameter of those polygons which have vertices at the remaining two points of intersection of the conics. Then the quantities y and η are connected by a symmetrical (2, 2) correspondence such that the values of y which give equal values for η are ± 1, $\pm\lambda^{-1}$. Therefore, if $y = \mathrm{sn}\,(u', \lambda)$, we must have $\eta = \mathrm{sn}\,(u' + \delta, \lambda)$, where δ is a constant.

We have arranged that y is divisible by x or $\mathrm{sn}\,u$, by making 0 the parameter of that polygon which has a vertex at the point 0. It appears thus that y must also be divisible by $x_1, x_2 \ldots x_{n-1}$, since, when any one of these is zero, y vanishes. We may write, therefore, $y = mxx_1 \ldots x_{n-1}$, where m is a constant, since y is only infinite when one or other of the x is infinite. The products $x_1 x_{n-1}$, $x_2 x_{n-2}$, &c., are given rationally as ratios of quadratic functions of x by the original equation of (2, 2) correspondence. To determine m, we have

$$y = m\,\mathrm{sn}\,u\,\mathrm{sn}\,(u+\gamma)\,\mathrm{sn}\,(u+2\gamma)\ldots\mathrm{sn}\,\{u+(n-1)\gamma\}\,(n\gamma = aK + biK')\,;$$

but, since $y=1$, when $x=\operatorname{sn} u=1$, or $u=K$, we have also

$$1=m \operatorname{sn} K \operatorname{sn}(K+\gamma) \operatorname{sn}(K+2\gamma) \dots \operatorname{sn}\{K+(n-1)\gamma\},$$

and therefore $$y=\frac{\operatorname{sn} u \operatorname{sn}(u+\gamma) \dots \operatorname{sn}\{u+(n-1)\gamma\}}{\operatorname{sn} K \operatorname{sn}(K+\gamma) \dots \operatorname{sn}\{K+(n-1)\gamma\}};$$

the denominator may, of course, be simplified, as in Jacobi's formula, but for present purposes it may be left as it stands. Now λ^{-1} is the value of y when $x=k^{-1}$, or when $u=K+iK'$,

$$\frac{\lambda}{k}=\frac{\operatorname{sn} K \operatorname{sn}(K+\gamma) \dots \operatorname{sn}\{K+(n-1)\gamma\}}{\operatorname{sn}(K+iK') \operatorname{sn}(K+iK'+\gamma) \dots \operatorname{sn}\{K+iK'+(n-1)\gamma\}},$$

which, again, may be easily reduced.

If we now assume that $u=Mu'$, where $y=\operatorname{sn}(u', \lambda)$ and M is a constant, we may determine M by observing that it is the value of $x : y$ when both of them are zero. Namely we have

$$M=\frac{\operatorname{sn}(K+\gamma) \operatorname{sn}(K+2\gamma) \dots \operatorname{sn}\{K+(n-1)\gamma\}}{\operatorname{sn}\gamma \operatorname{sn} 2\gamma \dots \operatorname{sn}(n-1)\gamma}.$$

And thus, with the exception of the assumption just made, the theory of transformation is established by means of that of the in-and-circumscribed polygon.

Let us now endeavour to generalize the theory of the in-and-circumscribed polygon. In the first place, we may observe that every involution of the third order gives rise to an elliptic transformation; for two triangles inscribed in a conic are always circumscribed to the same conic*. For convenience, we will now consider the involution as determined on the conic V; namely, the triangles form groups of three tangents which are in involution. And we may now generalize our theorem as follows:—

If a complete n*-gram move with its sides touching a conic so*

* Every substitution $y=\frac{U}{V}$, where U, V are cubic functions of x, has an elliptic differential which it transforms. The cubic forms U, V are first polars of two points in regard to a single quartic form F (Gundelfinger, *Math. Annalen*). Let $X=0$ give the four points whose first polars in regard to F have a square factor; then $dx : \sqrt{X}$ is the elliptic differential required. We have $X=jF+iH$, where H is the Hessian of F, and i, j the quadrinvariant and cubinvariant. [Cf. however p. 221, infra.]

that they form groups of n *tangents in involution, the locus of the* $\frac{1}{2}$ n (n − 1) *vertices is a curve of order* n − 1.

For, consider the number of points which the curve has in common with any one tangent of the conic. It determines uniquely in the involution the group of n tangents to which it belongs, and can have no other point on the locus of intersections except those $n-1$ in which it meets the other $n-1$ tangents of this group. Therefore &c.

If, in a curve of order n − 1, *it is possible to inscribe one complete* n*-gram whose sides all touch a conic, then it is possible to inscribe a singly infinite number, and the sides determine upon the conic an involution of the* nth *order.*

Let $A, B, C, \ldots N=0$ be the equations of the sides of the complete n-gram, then the equation of the curve may be written

$$\frac{\alpha}{A}+\frac{\beta}{B}+\frac{\gamma}{C}+\ldots+\frac{\nu}{N}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where $\alpha, \beta, \gamma \ldots \nu$ are constants. But since A is a tangent to the conic, its equation may be written in the form $X+aY+a^2Z=0$, where a is the parameter of the tangent and X, Y, Z three fixed lines. The condition for three tangents to meet in a point is

$$0=\begin{vmatrix} 1, & x, & x^2 \\ 1, & y, & y^2 \\ 1, & z, & z^2 \end{vmatrix} = (y-z)(z-x)(x-y) \ldots\ldots\ldots\ldots(2),$$

where x, y, z are their parameters. Hence the condition that the tangents whose parameters are x, y shall meet on the curve (1) is

$$\frac{\alpha}{(a-x)(a-y)}+\frac{\beta}{(b-x)(b-y)}+\ldots+\frac{\nu}{(n-x)(n-y)}=0 \ldots(3);$$

or, which is the same thing,

$$\Sigma\frac{\alpha}{a-x}=\Sigma\frac{\alpha}{a-y},$$

and therefore, if tangents xy and also xz meet on the curve, it follows that yz will meet on the curve. Starting, then, from any arbitrary tangent x, we can find the $n-1$ points in which

this meets the curve, and draw from them $n-1$ other tangents; the intersection of any two of these will lie on the curve, by what we have just proved. That is to say, we can inscribe a complete n-gram which shall have for one side any arbitrary tangent of the conic.

Now, suppose that x is given, then, regarding the equation (3) as determining $n-1$ values of y, we can find the product of the roots. Namely, it is

$$= \Sigma \frac{\alpha \,.\, \Pi a}{a\,(a-x)} \;:\; \Sigma \frac{\alpha}{a-x}, \quad \Pi a = a\,.\,b\,.\,c \ldots n.$$

If we multiply this product by x, we obtain the product of the parameters of all the tangents forming a complete n-gram; let this be called λ; then, observing that

$$\frac{\alpha x}{a\,(a-x)} = \frac{\alpha}{a} - \frac{\alpha}{a-x},$$

we shall find $$\left(1 - \frac{\lambda}{\Pi a}\right) \Sigma \frac{\alpha}{a-x} = \Sigma \frac{\alpha}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Now this is an equation of the nth order in x, the roots of which are the parameters of the sides of a complete n-gram; and λ is the product of these roots. Since λ is linearly involved, the equation shows that these groups of n tangents form an involution of the nth order, and that λ is proportional to the parameter of such a group in the involution when the groups containing the tangents 0, ∞ are made to have the parameters 0, ∞ respectively. It appears also that the sum of the roots, sum of their products in pairs, &c., are each given as linear functions of λ, and might each be used as parameters of the involution.

We shall now endeavour to find an expression for $ABC \ldots N$.

Let $1 - \lambda\,(\Pi a)^{-1} : \Sigma \alpha\, a^{-1}$ be called θ, and let $\Pi_a\,(x-a)$ mean the product $(x-a)\,(x-b) \ldots (x-n)$, then the equation for x may be put into the form

$$\Pi_a\,(x-a) + \theta \Pi_a\,(x-a)\,.\,\Sigma \alpha\,(x-a)^{-1} = 0 \quad \ldots\ldots\ldots (5),$$

where the first term is of order n, and the second of order

$n-1$ in x. By a slight change of notation, let the n roots be called $x_1, x_2, \ldots x_n$, and let $\Pi_x (y-x)$ mean

$$(y-x_1)(y-x_2)\ldots(y-x_n).$$

Then we have

$$(x-x_1)(x-x_2)\ldots(x-x_n) = \Pi_a (x-a) + \theta\Pi_a (x-a)\,\Sigma\alpha\,(x-a)^{-1};$$

and therefore

$$\Pi_x (y-x) = \Pi_a (y-a) + \theta\Pi_a (y-a)\,\Sigma\alpha\,(y-a)^{-1}.$$

Multiplying together two such equations, we obtain

$$\begin{aligned}\Pi_x (y-x) \,.\, \Pi_x (z-x) = &\\ &\Pi_a (y-a) \,.\, \Pi_a (z-a)\\ +\ &\theta\Pi_a (y-a) \,.\, \Pi_a (z-a) \,.\, \{\Sigma\alpha\,(y-a)^{-1} + \Sigma\alpha\,(z-a)^{-1}\}\\ +\ &\theta^2\Pi_a (y-a) \,.\, \Pi_a (z-a) \,.\, \Sigma\alpha\,(y-a)^{-1} \,.\, \Sigma\alpha\,(z-a)^{-1}.\end{aligned}$$

Now, if we examine this equation, we shall find that the left-hand member is $A'B'C'\ldots N'$, where $A'=0$, $B'=0\ldots$ are the tangents which make up the n-gram belonging to the parameter θ; the first term on the right is $ABC\ldots N$, the n-gram for $\theta=0$, and the last term is $\theta^2 Z_1 Z_2 \ldots Z_n$, the n-gram for $\theta=\infty$, which has the line $Z=0$ in it. Thus we may write the equation

$$\Pi A' = \Pi A + \theta P_n + \theta^2 \Pi Z \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

and it only remains to find the nature of the curve P_n of the nth order. We may see from its equation,

$$0 = \Pi_a (y-a)\,\Pi_a (z-a)\,\{\Sigma\alpha\,(y-a)^{-1} + \Sigma\alpha\,(z-a)^{-1}\},$$

that it passes through the points of contact and all the intersections of the n tangents ΠA; and then it is clear, from the symmetry, that it must pass through the points of contact and all the intersections of the tangents ΠZ. But perhaps the simplest way is to consider the envelope of the n-gram $\Pi A'$, which we know must consist of the conic K_2 once, and the locus of the nodes C_{n-1} twice; thus we shall have

$$4\Pi A \,.\, \Pi Z - P_n^2 = K_2 C_{n-1}^2$$

to a factor *près*, and this equation gives at once the intersections of P_n with K_2, and with the n-gram.

We may now state the following propositions :—

Given any two in-and-circumscribed n-*grams* ΠX *and* ΠZ, *there exists always a curve* P_n *of order* n *which passes through their* $n(n-1)$ *vertices and their* 2n *points of contact with the conic.*

The equation of any other n-*gram may be written in the form*

$$0 = \Pi X + \lambda P_n + \lambda^2 \Pi Z.$$

The relation between λ, *the parameter of the* n-*gram, and* x *the parameter of one of its sides, is*

$$(\Pi a - \lambda)\, \Sigma \alpha\, (a - x)^{-1} = \Pi a \,.\, \Sigma \alpha a^{-1},$$

and λ *is the product of the roots of this equation.*

I have here taken ΠX, $(X_1 X_2 \dots X_n)$ for the first n-gram, corresponding to $\lambda = 0$, instead of ΠA, corresponding to $\lambda = \Pi a$ or $\theta = 0$.

We may show, conversely, that if the envelope of

$$0 = P + \lambda Q + \lambda^2 R,$$

where $P = 0$, $Q = 0$, $R = 0$ are three curves of the nth order, is $4PR - Q^2 = K_2 C^2$, C being of order $n-1$; then P and R are each an assemblage of n straight lines. For the curve P has nodes on all its intersections with C, since $4PR = Q^2 + K_2 C^2$; that is, $\frac{1}{2}n(n-1)$ nodes, so that it must consist of n straight lines.

[This point of view immediately suggests the extension of the whole theory to quadric surfaces. If the envelope of $P + \theta Q + \phi R + \theta\phi S$ is $PS - QR = K_2 C^2$, where P, Q, R, S are of order n, and C of order $n-1$, each of the surfaces P, Q, R, S will meet the quadric K_2 in two curves of the nth order, and therefore will have $\frac{1}{2}n^2$ contacts with it; and similarly will meet C_{n-1} in two curves of order $\frac{1}{2}n(n-1)$, which intersect in $\frac{1}{4}n^2(n-1)$ points; these are not contacts, but nodes on the surface P. We thus get a theory of surfaces of order n, having $\frac{1}{2}n^2$ contacts with a quadric surface, and $\frac{1}{4}n^2(n-1)$ nodes on a fixed surface of order $n-1$. It appears that n must be even, and of course the variable surface is subject to other conditions.

Thus, in the case of a cone doubly tangent to the quadric, and having its vertex on a fixed plane, it has also to pass through two fixed points on the plane and four on the quadric.

The application of the identity $4PR - Q^2 = K_2 C^2$ to surfaces only reproduces the theory of the plane conic.}

At any point of intersection of the two n-grams,

$$0 = \Pi X + xP_n + x^2 \Pi Z,$$

$$0 = \Pi X + yP_n + y^2 \Pi Z,$$

we shall have $xy : -x-y : 1 = \Pi X : P_n : \Pi Z$.

Consequently, any symmetrical (m, m) correspondence between the two n-grams expresses that they intersect on a curve of order mn, namely $(\Pi X, -P_n, \Pi Z)^m$, if the equation of the (m, m) correspondence is $(xy, x+y, 1)^m$.

If we substitute the values $xy : -x-y : 1$ for $\Pi X : P_n : \Pi Z$, in the equation of a third n-gram $0 = \Pi X + zP_n + z^2 \Pi Z$, we shall get simply $(z-x)(z-y) = 0$. Consider, then, m different n-grams ΠA_1, $\Pi A_2 \ldots \Pi A_m$, and form the curve of order $n(m-1)$

$$\frac{\beta_1}{\Pi A_1} + \frac{\beta_2}{\Pi A_2} + \ldots = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

If the n-grams x, y meet on this curve we shall have

$$\Sigma \frac{\beta}{(a-x)(a-y)} = 0;$$

or, what is the same thing,

$$\Sigma \frac{\beta}{a-x} = \Sigma \frac{\beta}{a-y},$$

where $a_1, a_2 \ldots a_m$ are the parameters of the n-grams. It follows, as before, that a singly infinite number of groups of n-grams can be totally inscribed in the curve (7); a group being totally inscribed when all the intersections of any two n-grams of the group are on the curve.

So far we have dealt only with totally inscribed n-grams; and, as this case is represented only by the triangle when the curve of inscription is a conic, it might seem that there should

be more general theorems corresponding to the inscription of polygons of a greater number of sides in a conic. But, in fact, the case of total inscription is the general case, and all others are cases of decomposition of the curve C_{n-1}. Consider, for example, a hexagon inscribed in a conic. If we produce all the sides, we shall get nine more intersections; three of these lie on a straight line, and the other six on a conic which circumscribes two triangles. These two conics and the straight line make up the curve C_5 of the fifth order. Again, let eight lines A, B, C, D, F, G, H, K (fig. 30) touch a conic, and let the twelve points marked o in the figure touch a cubic; then the octagram may be moved round the conic so as to keep these twelve points on the cubic. The points marked × will lie on a fixed straight line, and a second cubic will pass through the intersections of A, B, C, D among themselves, and of F, G, H, K among themselves. These two cubics and the straight line make up the curve C_7 of the seventh order; and it is easy to see that there is an analogous case for any even number of lines. In order that a porismatic polygon may be inscribed in a curve, it is necessary that either the order or the curve or the number of sides should be even.

XXIII.

NOTES ON THE COMMUNICATION ENTITLED "ON THE TRANSFORMATION OF ELLIPTIC FUNCTIONS."*

SOME of the following notes† would have been incorporated in the paper by the process of revision for the press, if that had not been kindly performed for me during an enforced absence from the English climate. As regards all but one of them, I am glad of the opportunity which has thus been afforded me of extension and correction. But it is a matter of great regret to me that I discovered too late the priority of M. Darboux in the principal theorem of the second part of the paper; viz., the porismatic character of a polygram circumscribed to a conic and totally inscribed in a curve of order one less than the number of sides. In one of the notes to a book which it is almost inexcusable in a geometer not to have read, marked, learned, and inwardly digested‡, M. Darboux has stated and proved the theorem, and has followed it by further investigations of the highest interest and importance. The method even of my investigation is the same as that of M. Darboux (as indeed was inevitable from the nature of the subject), namely, the representation of a point in a plane by means of the parameters of the tangents drawn from it to a fixed conic. It is not the first time

* [From the *Proceedings of the London Mathematical Society*, Vol. VII. Nos. 102, 103, pp. 225—233.]

† These remarks apply also to certain developments which I have since thought it better to communicate under separate titles.

‡ *Sur une classe remarquable de courbes et de surfaces algébriques.* Paris, Gauthier-Villars, 1873. Note II., p. 183.

that I have had the honour of following, however distantly, in the footsteps of that eminent geometer; but on other occasions it has been my good fortune to discover the fact in time.

Completion of the Geometric Proof of the Transformation-Formulæ.

In my former paper one point was assumed as given by the analytical theory of transformation, namely, that the new argument u' is equal to the old argument u divided by a constant M. Having now found a simple proof of this, I will take the liberty of re-stating in outline, for the sake of clearness, the whole demonstration; availing myself of a remark of M. Darboux.

It is proved by Jacobi's method that if $x_1\ x_2 \ldots x_n$ are parameters of the points of contact of the n sides of a polygon circumscribed to a conic U and inscribed in a conic V, and if ± 1, $\pm k^{-1}$ are the parameters of the points of contact of the common tangents of the two conics, then

$$x_1 = \mathrm{sn}\, u,\ x_2 = \mathrm{sn}\,(u+\gamma),\ x_3 = \mathrm{sn}\,(u+2\gamma), \ldots x_n = \mathrm{sn}\,\{u+(n-1)\gamma\},$$

where $n\gamma = 4K + 4iK'$, and the modulus of the elliptic function is k.

This being so, an infinite number of in-and-circumscribed polygons can be drawn.

If $p_n = 0$ be the equation in x whose roots are $x_1\ x_2 \ldots x_n$, and $q_n = 0$ an equation in x whose roots are the parameters of the sides of another such polygon; then $p_n - yq_n = 0$ will have for its roots the parameters of the sides of an in-and-circumscribed polygon, whatever value be given to y.

For the locus of the $\frac{1}{2}n\,(n-1)$ intersections of the tangents at the n points $p_n - yq_n = 0$, when y is made to vary, is a curve of order $n-1$, which has $2n$ points in common with the conic V, and therefore contains that conic entirely*.

The quantity y being now regarded as the parameter of a varying polygon, let p_n be chosen to represent that polygon

* This is in substance the remark of M. Darboux referred to. *Op. cit.*, p. 190.

which has the parameter of one of its sides equal to zero, and q_n to represent that which has the parameter of one of its sides infinite. Then y, which is $p_n : q_n$, will be an odd function of x, because of the symmetry of the figure and the fact that $q_n = 0$ (having one root infinite) is only of degree $n-1$.

If then p_n and q_n be affected with such constant multipliers that $y=1$ when $x=1$, we must have $y=-1$ when $x=-1$. And we may suppose that $y = \pm\lambda^{-1}$ when $x = \pm k^{-1}$.

Now the intersections of corresponding sides of two in-and-circumscribed polygons lie on a conic touching the common tangents of U and V.

For the parameters being respectively sn u, sn $(u+\gamma)$, &c., and sn v, sn $(v+\gamma)$, &c., the common difference of the arguments is $u-v$.

If we suppose the two polygons to vary subject to this condition, their parameters y and η will be connected by a (2, 2) correspondence, such that the values of y which make the two corresponding values of η coincide are ± 1, $\pm\lambda^{-1}$.

Therefore, if $y = \text{sn}\,(u', \lambda)$, we must have $\eta = \text{sn}\,(u'+c', \lambda)$, where c' is a constant. But the relation between corresponding sides x, ξ, of these polygons is $x = \text{sn}\,(u, k)$, $\xi = \text{sn}\,(u+c, k)$, since they intersect on the fixed conic V.

Hence the quantities u, u' are so related that a constant difference between two values of u implies a constant difference between the corresponding values of u'. Hence* (by Euclid's definition of proportion) a varying difference between two values of u implies a *proportional* difference between the corresponding values of u'. But $u'=0$ when $u=0$; therefore the two quantities are proportional, and $u = Mu'$ where M is constant.

Now the relation between the parameters of two consecutive sides of a polygon being of the second degree in each, the products x_2x_n, x_3x_{n-1}, &c., are given as ratios of quadratic functions of x_1. Hence the product $x_1x_2 \ldots x_n$, regarded as a function of

* This is Archimedes' proof that a body which passes over equal spaces in equal times will pass over proportional spaces in unequal times.

x_1, is a rational fraction whose numerator is of order n, and whose denominator is of order $n-1$. But y is a rational fraction whose numerator and denominator are of just these orders; and y vanishes whenever one of the quantities $x_1, x_2 \ldots x_n$ vanishes, and becomes infinite when one of them becomes infinite. Therefore

$$y = mx_1x_2x_3 \ldots\ldots x_n,$$

where m is a constant; that is to say,

$$\text{sn}\left(\frac{u}{M}, \lambda\right) = m \text{ sn } u \text{ sn } (u+\gamma) \text{ sn } (u+2\gamma) \ldots\ldots \text{ sn } \{u+(n-1)\gamma\},$$

when $n\gamma = 4K + 4iK'$. We determine m by remarking that $y=1$, when $x=1$, or when $u=K$; and then λ, by remarking that $y=\lambda^{-1}$, when $x=k^{-1}$, or when $u=K+iK'$. Finally M is determined by differentiating the equation and making $u=0$.

(*Cayley's Theorem.*)—*Every Cubic Transformation has an Elliptic Differential which it transforms.*

This theorem was given by Prof. Cayley in the *Philosophical Magazine*, Vol. 15 (Fourth Series), p. 363. I here reproduce his investigation, slightly altered to suit the generalization which follows. On the very beautiful solution of the complete question (Given the elliptic differential, to find the transformation) by Hermite (*Crelle*, vol. 60, p. 304) and Clebsch (*Theorie der binären alg. Formen*, p. 405), I hope to say something at another time*.

Let U, V be any two cubic functions of x, and consider the transformation $y = \frac{U}{V}$.

Suppose that

$$\text{Disct. } (U - Vy) = A + By + Cy^2 + Dy^3 + Ey^4,$$

where, of course, $A = \text{Disct. } U$, $E = \text{Disct. } V$. And let y_1, y_2, y_3, y_4

* The foot-note in my previous paper gave an erroneous expression for X. The article there referred to (Gundelfinger, *Math. Annalen*, Vol. VII., p. 452) is a simplification of the method and results of Clebsch in regard to the typical representation of two cubics.

be the roots of the equation Disct. $(U - Vy) = 0$. Then each of the cubics $U - Vy_1$, $U - Vy_2$, $U - Vy_3$, $U - Vy_4$ has a square factor, because its discriminant vanishes. Now, if $U - Vy_1$ has a square factor $(x - \alpha)^2$, then $x - \alpha$ divides $U' - V'y_1$; that is, for the value $x = \alpha$ we have at the same time

$$\begin{matrix} U - Vy_1 = 0 \\ U' - V'y_1 = 0 \end{matrix} \qquad \left\{ U', V' = \frac{dU}{dx}, \frac{dV}{dx} \right\},$$

and therefore also $VU' - V'U = 0$; that is to say, those four linear factors which are squared in the cubics $U - Vy_1$, &c., occur as single factors in $VU' - V'U$. It follows that

$$A(U - Vy_1)(U - Vy_2)(U - Vy_3)(U - Vy_4) = (VU' - V'U)^2 . X,$$

where X represents the product of the single factors of the four cubics. Or, which is the same thing,

$$AU^4 + BU^3V + CU^2V^2 + DUV^3 + EV^4 = (VU' - V'U)^2 X.$$

Now, since $y = \frac{U}{V}$, we have $dy = \frac{VU' - V'U}{V^2} dx$. Hence, if we transform the differential $\frac{dy}{\sqrt{\text{Disct.}(U - Vy)}}$ by the substitution $y = \frac{U}{V}$, we get

$$\frac{VU' - V'U}{V^2} \cdot \frac{V^2}{(VU' - V'U)\sqrt{X}} dx \text{ or } \frac{dx}{\sqrt{X}}.$$

That is, we have $$\frac{dy}{\sqrt{\text{Disct.}(U - Vy)}} = \frac{dx}{\sqrt{X}},$$

where y is connected with x by the equation $y = \frac{U}{V}$, which is the theorem in question.

New Stand-Point for the Algebraic Transformation-Theory.

We may generalize this result by applying an analogous treatment to transformations of any order. The problem is considered in the following form: Given a transformation $y = U : V$, it is required to find—

(1) What are the necessary and sufficient conditions to be satisfied by the functions U, V, in order that the transformation $y = U : V$ may be able to transform an elliptic differential;

(2) These conditions being supposed satisfied, what is the differential which can thus be transformed.

We will consider first the case in which U and V are of odd order, say $2m + 1$, or, to speak more correctly, the case in which $U - Vy$ is of order $2m + 1$ in x.

The necessary conditions may at once be derived from consideration of the varying in-and-circumscribed polygon the parameters of whose sides are the roots of the equation $U - Vy = 0$.

Starting with any one side of the polygon, which touches what we may call the inner conic, we find its intersections with the outer conic, and then from these draw new tangents to the inner conic. Proceeding in this way symmetrically on both sides of the original tangent, we find at last that the two tangents to the inner conic meet on the same point of the outer conic. We must clearly end with two tangents, and not with one, because the polygon has an odd number of sides.

We might, however, start with a vertex of the polygon on the outer conic, draw two tangents to the inner conic, then from their intersection with the outer conic two more tangents, and so on: at last we shall reach a pair of vertices such that the line joining them touches the inner conic. In this case we must end with two vertices on one side, not with one vertex on two sides, for the same reason as before, that we have supposed the polygon to have an odd number of sides.

Now suppose that in the first mode of construction we start with a common tangent to the two conics; then its two intersections with the outer conic will coincide, and consequently the tangents from them to the inner conic coincide also. We may, however, go on with the construction; and, after drawing $m + 1$ successive tangents, we shall have an exceptional case of an in-and-circumscribed polygon, in which the side first drawn (the common tangent of the two conics) counts singly, and each

of the m others counts doubly, so that the polygon has altogether $2m + 1$ sides. But the last pair of tangents being coincident, must be regarded as intersecting on the inner conic; and therefore their point of contact must be an intersection of the two conics.

So that we cannot by the second mode of construction get a degenerate polygon of an odd number of sides different from those just considered. If we start with a vertex at a point of intersection of the two conics, the tangents drawn from this to the inner conic will of course coincide, and so therefore will the points in which they meet the outer conic again, and we may continue the process; but we only have a right to stop when the line joining the two last coincident vertices on the outer conic (*i. e.*, the tangent at that point to the outer conic) touches the inner conic; that is to say, when we come upon a common tangent of the two conics.

What actually happens may be illustrated by the case of an in-and-circumscribed pentagon. Let ab be a common tangent of the two conics, touching the inner conic at a, and the outer at b. From b draw the other tangent bc to the inner conic, meeting the outer conic again at c; then from c draw the other tangent cd to the inner conic meeting the outer conic again at d. Then, *if pentagons can be drawn inscribed to the outer conic, and circumscribed to the inner, the point* d *will be an intersection of the two conics.* And the pentagon whose sides are dc, cb, bab, bc, cd is a degenerate case of an in-and-circumscribed pentagon; the side bab being single, and the sides bc, cd, each of them double.

Observe that, if d were not an intersection of the two conics, we should still have an improper solution of the problem, to find five points on the outer conic such that the line joining every successive two shall touch the inner conic. But if we started from d as an intersection of the conics, and then found the points c, b, a as before, except that ba is now not a tangent to the outer conic at b, we should not have found a solution of that problem, but of this other—To find five tangents to the inner conic, so that the intersection of every successive two shall be upon the outer conic.

For the purpose of our present investigation, the result may be stated thus: an in-and-circumscribed polygon of an odd number of sides can only have two sides coincident, when one of its sides is a common tangent of the two conics, and all the others coincide in pairs.

Or, the equation $U - Vy = 0$ can only have coincident roots, when one represents a common tangent, and the others coincide in pairs: so that $U - Vy$ has one single factor, and m square factors. Moreover, there are four values of y, and four only, which bring this about. Now the discriminant of $U - Vy$ is of the order $4m$. Hence *we must have*

$$\text{Disct. } (U - Vy) = Y^m, \text{ where } Y = (1, y)^4,$$

and if $y_1 y_2 y_3 y_4$ *are the roots of the equation* $(1, y)^4 = 0$, *then each of the quantities* $U - Vy_1$, &c., *has one single factor and m square factors.*

These conditions are necessary; we shall shew that they are sufficient, by solving the second part of the problem.

In the first place, we have

$$(1, 0)^4 . (U - Vy_1)(U - Vy_2)(U - Vy_3)(U - Vy_4) = (U, V)^4,$$

and therefore, as before,

$$(U, V)^4 = (VU' - V'U)^2 X,$$

where $(VU' - V'U)^2$ is the product of all the square factors, and X of the four single factors. From this it follows immediately that

$$\frac{dy}{\sqrt{Y}} = \frac{dx}{\sqrt{X}},$$

if $y = \frac{U}{V}$; which is the transformation required. The result may thus be stated:—

The necessary and sufficient conditions that the substitution $y = U : V$ shall be able to transform an elliptic differential, $U - Vy$ being of order $2m + 1$ in x, are that Disct. $(U - Vy)$ shall be a perfect mth power, and that those forms $U - Vy$ which have a square factor at all shall have m square factors. This

being so, the differential $\frac{dy}{\sqrt[m^2]{\text{Disct. } (U - Vy)}}$ will be transformed by the given substitution into $\frac{dx}{\sqrt{X}}$, where X is the product of the four single factors.

It may be observed that, if those forms $U - Vy$ which have a square factor at all have m square factors, it will follow that Disct. $(U - Vy)$ is a perfect mth power; but the converse is not true.

Passing now to the case of a transformation of even order, we enquire, as before, in what cases the in-and-circumscribed polygon can have two sides coincident. If we start with a tangent to the inner conic, and from its intersections with the outer conic draw two more tangents, and so on; there cannot be an in-and-circumscribed polygon of an even number of sides, unless we come to a pair of intersections such that the line joining them touches the inner conic. Suppose then that the first side is a common tangent, so that its two intersections with the outer conic coincide, and that we draw another tangent from this point, another from its second point of intersection, and so on; we must finally come to a point on the outer conic where the tangent touches the inner conic; that is, we must come to another common tangent. In the case of a quadrilateral, for example, let ab be a common tangent, touching the inner conic at a and the outer at b. From b draw the other tangent to the inner conic, meeting the outer again at c; then cd must be also a common tangent, touching the outer conic at c and the inner at d. Thus the sides of the degenerate quadrilateral are bab, bc, cdc, cb, the sides bab, cdc counting singly, and bc double. And in general the two common tangents will count singly, and all the rest double. It is manifest that there are only two degenerate polygons of this kind, each containing two of the four common tangents.

The second construction also gives us two degenerate polygons, but of quite a different character. Starting with a point on the outer conic, we draw two tangents to the inner, and from their new intersections with the outer, two more, and so on;

we must at last come to two tangents which meet on the outer conic. If then our starting-point is a point of intersection, so that the two tangents coincide all through the process, we must come to a pair whose intersection, that is, their point of contact with the inner conic, is on the outer conic; or, which is the same thing, we must come to another intersection of the conics. To use again the quadrilateral as an illustration, the tangents to the inner conic at two points of intersection α and γ must meet on the outer conic at β, and the sides of the quadrilateral are then $\alpha\beta$, $\beta\gamma$, $\gamma\beta$, $\beta\alpha$, so that *all* of them count double. And generally, in degenerate polygons of this kind, all the sides count double. There are clearly two such degenerate polygons, each having two points of intersection for vertices.

To sum up, then, there are four degenerate polygons of even order $2m$; two of them have each two common tangents as sides, and two of them have each two points of intersection as vertices. The former have the common tangents as single sides, and all the rest double; the latter have all their sides double.

It follows that, if the substitution $y = U : V$ is capable of transforming an elliptic differential, $U - Vy$ being of order $2m$ in x, there are only four values of y which make $U - Vy$ have a square factor; two of these make it have m square factors, and the other two make it have $m-1$ square factors and two single factors. Consequently the former two are m-fold roots, and the latter two $(m-1)$-fold roots, of the equation $\text{Discr.}(U - Vy) = 0$. That is to say, we have

$$\text{Discr.}(U - Vy) = Y^{m-1} \cdot (y - y_1)(y - y_2),$$

where $$Y = k(y - y_1)(y - y_2)(y - y_3)(y - y_4) = (1, y)^4, \text{ say.}$$

Hence, as before, $$(U, V)^4 = (VU' - V'U)^2 \cdot X,$$

where X is the product of the four single factors due to y_3 and y_4. From these equations it follows directly that

$$\frac{dy}{\sqrt{Y}} = \frac{dx}{\sqrt{X}},$$

which is the transformation required.

In regard to these conditions it is to be observed that in general they imply a special constitution of the quantics U, V,

as well as a special relation of them to each other. This consideration, however, does not come in until the sixth order of transformation is reached. Thus, in the case of the quartic transformation, the only condition is that U, V shall be simultaneously reducible to the canonical form; which being so, we may find linear combinations of them such that one is the Hessian of the other, thus falling back upon Hermite's very elegant form. In the quintic transformation U may be taken arbitrarily, but the involution $U - Vy$ is then completely determined. In the sextic transformation, however, U and V must each be the product of three quartics in involution (viz., the same involution in the two cases); so that a certain invariant of each must vanish. (Salmon's *Higher Algebra*, p. 210 and Appendix; see Clebsch, *Alg. Formen*, p. 298.)

*ADDITIONS TO PAPER ON THE TRANSFORMATION OF ELLIPTIC FUNCTIONS.

1. Completion of doctrine of in-and-circumscribed polygon so as to make it a formal proof of the transformation theory. In p. [210] the quantities u, u' are proportional because constant difference in the values of u means constant difference in the values of u', and they vanish together.

2. Porismatic representation of spherical harmonics as sums of sectorial harmonics. "Nodal curve" of order n may be expressed as a sum of nth powers, $n+1$ in number in a singly infinite number of ways.

3. Polyhedra whose faces osculate a skew-cubic. Special case noticed by Frahm. Involution of one variable: curve and ruled surface. Involution of two variables, surface of order $n-2$; (2,2,2) correspondence for quadric surfaces; cases of degeneration: Δ-faced polyacra. Application of Cotterill's theorem; vanishing area and volume of the porismatic polygon and polyacron.

4. Multiplication. Sylvester's theory of derived points on cubic. Do. for quadriquadric. Scrolls. Arithmetical Theorem.

* [These mere heads of an intended paper are printed as they were found: no complete paper seems to embody them. Perhaps 1 was worked up into * XXIV., and 2 seems to be connected with XXV.]

Fig. 23.

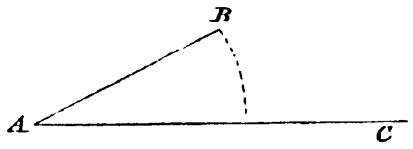
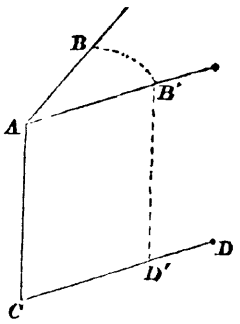


Fig. 24.



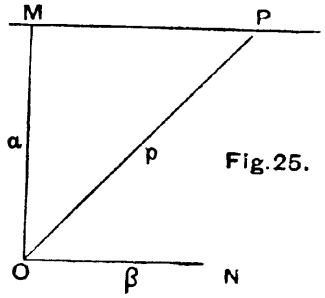


Fig. 25.

Fig. 26.

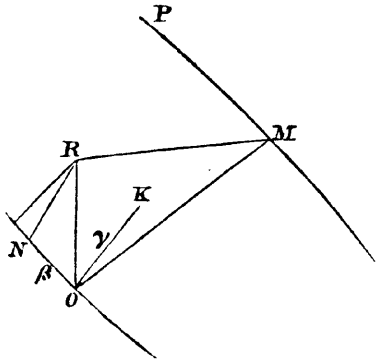


Fig. 27.

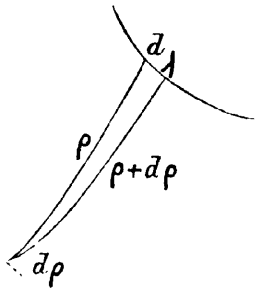


Fig. 28.

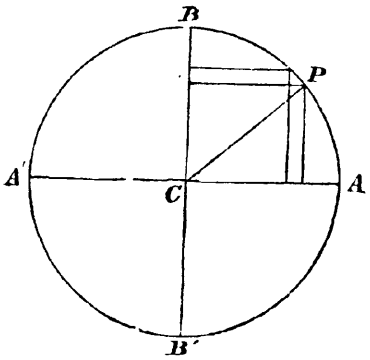


Fig. 29.

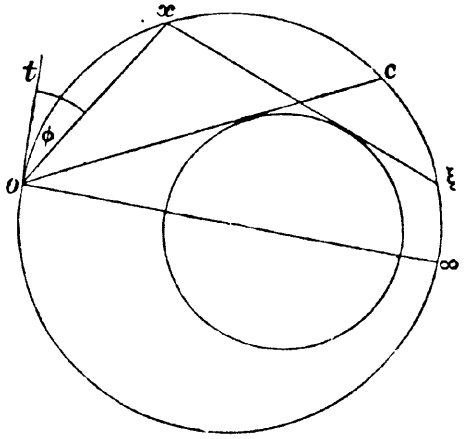
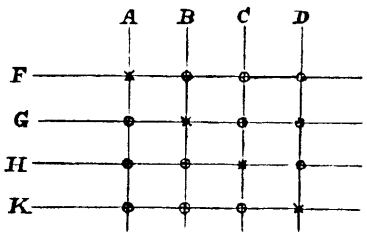


Fig. 30.



face p. 228.

*XXIV.

ON IN-AND-CIRCUMSCRIBED POLYHEDRA.

THE extension of the theory of the in-and-circumscribed polygon, made in my paper "On the Transformation of Elliptic Functions," was suggested by a particular case of it studied by Dr. Lüroth in the *Math. Annalen*, I. [pp. 37—53]. It had been remarked by Clebsch (*Crelle*, [Bd. 59]) that not every curve of the fourth order can have its equation expressed as the sum of five fourth-powers; but that for this to be possible a certain invariant (the determinant of the six second derived forms) must vanish. Dr. Lüroth proved that in this case the proposed reduction can be effected in a singly infinite number of ways, and that the lines represented by equating the fourth-powers to zero all touch the same conic. All these pentagrams, moreover, are totally inscribed in a covariant quartic of the original curve (locus of points whose covariant cubics are equianharmonic, so that their Hessians break up into three straight lines). We thus have a varying pentagram circumscribed to the conic and totally inscribed in a quartic; and it was this pentagram which suggested the theorems in my paper above referred to.

The starting-point of the present communication is a remark by Dr. Frahm on a question allied to the foregoing (*Math. Ann.* VII. p. 635). It had been assumed by Dr. Salmon that the equations of three quadric surfaces may simultaneously be reduced to the form of the sum of five squares. Now Hesse

established a connection between the theory of three quadric surfaces and that of a plane quartic curve; namely, if u, v, $w = 0$ are the three surfaces, then the equation

$$\text{Discr. } (\lambda u + \mu v + \nu w) = 0$$

is of the fourth order in λ, μ, ν, and taking these as co-ordinates of a point in a plane, the equation is that of a quartic curve which corresponds point for point with the locus of vertices of the cones $\lambda u + \mu v + \nu w = 0$. Dr. Frahm remarked that if the three quadrics could be simultaneously reduced to Salmon's canonical form, then this quartic curve is totally circumscribed to a pentagram, and is therefore not the general quartic, but the special form pointed out by Lüroth; so that the problem of effecting this reduction is porismatic—it can either not be solved at all or be solved in a singly infinite number of ways. In the latter case, then, we have a singly infinite number of pentaplanes in regard to which the reduction can be effected; and each of these is totally inscribed in a curve of order 6 and deficiency 3, locus of the vertices of the cones

$$\lambda u + \mu v + \nu w = 0.$$

On considering the envelope of these pentaplanes, I found it to be a twisted cubic. The road to further generalizations was now clearly open.

I.

The equation of any osculating plane to a twisted cubic may be written

$$X + 3\theta Y + 3\theta^2 Z + \theta^3 W = 0,$$

where X, Y, Z, $W = 0$ are four fixed planes, and θ a parameter determining the particular osculating plane. The developable generated by tangent lines to the cubic is

$$(XZ - Y^2)(YW - Z^2) - 4(XW - YZ)^2 = 0 \quad \ldots\ldots\ldots(1),$$

and from this we see that Y passes through the tangent line of X and the point of contact of W, Z through the tangent line of W and the point of contact of X; or we may say that XY

and ZW are two tangent lines, YZ their chord of contact. The equations to the cubic itself are

$$\begin{vmatrix} X, & Y, & Z \\ Y, & Z, & W \end{vmatrix} = 0.$$

The co-ordinates of the point of intersection of three planes x, y, z are

$$\begin{vmatrix} 1, & 3x, & 3x^2, & x^3 \\ 1, & 3y, & 3y^2, & y^3 \\ 1, & 3z, & 3z^2, & z^3 \end{vmatrix} = 3xyz : -yz - zx - xy : x + y + z : -3,$$

and if we substitute these in the function $X + 3aY + 3a^2Z + a^3W$ belonging to any fourth plane, we get $3(x-a)(y-a)(z-a)$.

If a variable group of n *osculating planes of a twisted cubic form an involution of the* n^{th} *order, the locus of their lines of intersection is a ruled surface* R *of order* $2(n-1)$, *and the locus of their points of intersection is a curve* γ *the order of which is* $\frac{1}{2}(n-1)(n-2)$, *and which is a triple curve on the ruled surface.*

Consider the sections of this curve and surface by a fixed osculating plane L of the cubic. There is one group of the involution to which it belongs, and it meets the curve only where it meets the lines of intersection of the remaining $n-1$ planes of this group; that is, in $\frac{1}{2}(n-1)(n-2)$ points. This therefore is the order of the curve. All other osculating planes meet the plane L in lines which touch a fixed conic in that plane; in fact it meets the developable (1) in this conic and in its tangent line taken twice. The variable group of n planes in involution determines upon L a variable group of n tangents in involution; and the locus of their intersections is a curve of order $n-1$, by what we have already proved. Besides this curve, the plane L meets the ruled surface in $n-1$ straight lines, in which it meets the $n-1$ other planes of the group which contains it; so that the order of the whole intersection is $2(n-1)$, which is therefore the order of the surface. That the curve is a triple curve on the surface is clear from the fact that through any point of it there may be drawn three straight lines in the

surface, these being symmetrically related and not in the same plane.

By means of the in-and-circumscribed polygon determined upon the plane L we may find the equation of the ruled surface. Since the parameters of the several osculating planes of the cubic may be taken as parameters of the tangents to the conic in the plane L in which they are cut by that plane, it follows that the condition for two osculating planes x, y to belong to the same group is the same as the condition for the two tangents x, y to belong to the same group; that is to say, it is a condition of the form

$$\Sigma_i \frac{\alpha_i}{(x-a_i)(y-a_i)} = 0 \text{ or } \Sigma_i \frac{\alpha_i}{x-a_i} = \Sigma_i \frac{\alpha_i}{y-a_i} \ldots\ldots\ldots (2),$$

where $a_1, a_2, \ldots a_n$ are parameters of some one group, and $\alpha_1, \alpha_2, \ldots \alpha_n$ are constants.

When this condition is fulfilled, the line of intersection of the planes x, y meets the curve γ. Now through any point on the surface R there can be drawn a line which is the intersection of two osculating planes of the cubic, and these two planes will satisfy the condition (2). If therefore the point of intersection of xyz lies on the surface R, the condition (2) must be satisfied either for yz or for zx or for xy. That is, we must have

$$\Sigma \frac{\alpha_i}{(y-a_i)(z-a_i)} \cdot \Sigma \frac{\alpha_i}{(z-a_i)(x-a_i)} \cdot \Sigma \frac{\alpha_i}{(x-a_i)(y-a_i)} = 0 \ldots\ldots (3).$$

It remains to translate this equation into the ordinary point-co-ordinates. Let $A_1, A_2, \ldots A_n = 0$ be the equations to the n planes whose parameters are $a_1, a_2, \ldots a_n$; that is, let

$$A_i = X + 3a_iY + 3a_i^2Z + a_i^3W;$$

then the result of substituting in A_i the co-ordinates of the point of intersection of x, y, z is $3(x-a_i)(y-a_i)(z-a_i)$. If then we multiply together terms of like suffixes in the factors of (3), we get in the product the sum of n terms

$$9\Sigma_i \frac{\alpha_i^3}{A_i^2}.$$

Next, the equation of the plane passing through the tangent line of A_i and the point of contact of A_j is

$$(iij) = X + 2a_i Y + a_i^2 Z + a_j (Y + 2a_i Z + a_i^2 W) = 0,$$

and when we substitute in this the co-ordinates of intersection of xyz, we obtain

$$(x-a_j)(y-a_i)(z-a_i) + (y-a_j)(z-a_i)(x-a_i) + (z-a_j)(x-a_i)(y-a_i).$$

Selecting then from (3) the products of two like suffixes by one unlike, we get the sum of $n(n-1)$ terms

$$\Sigma_{ij} \frac{\alpha_i^2 \alpha_j \,.\, (iij)}{A_i^2 \,.\, A_j}.$$

Lastly the equation of the plane passing through the points of contact of A_i, A_j, A_k is

$$(ijk) = X + (a_i + a_j + a_k)\, Y + (a_j a_k + a_k a_i + a_i a_j)\, Z + a_i a_j a_k\, W = 0,$$

and when we substitute in this the co-ordinates of intersection of x, y, z, we get $\frac{1}{2}\Sigma (x-a_i)(y-a_j)(z-a_k)$, where the x, y, z are to be permuted in all possible ways. Thus the products in (3) of three unlike suffixes give us the $\frac{1}{6} n(n-1)(n-2)$ terms

$$\tfrac{1}{2} \Sigma_{ijk} \frac{\alpha_i \alpha_j \alpha_k \,.\, (ijk)}{A_i A_j A_k},$$

and the equation to the surface R is therefore

$$0 = 18\Sigma_i \frac{\alpha_i^3}{A_i^2} + 2\Sigma_{ij} \frac{\alpha_i^2 \alpha_j \,.\, (iij)}{A_i^2 A_j} + \Sigma_{ijk} \frac{\alpha_i \alpha_j \alpha_k \,.\, (ijk)}{A_i A_j A_k}.$$

The equation shews that when cleared of fractions it is as it ought to be of the order $2(n-1)$.

XXV.

ON A CANONICAL FORM OF SPHERICAL HARMONICS*.

THE canonical form in question is an expression of the general harmonic of order n as the sum of a certain number of sectorial harmonics, this number being, when n is even,

$$\frac{5n-10}{2},$$

and when n is odd,

$$\frac{5n-9}{2}.$$

Laplace's operator,

$$\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2},$$

may be obtained from the tangential equation of the imaginary circle $\xi^2+\eta^2+\zeta^2=0$, by substituting $\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{d}{dz}$ for ξ, η, ζ. If, therefore, a form $U\equiv(x, y, z)^n$ is reduced to zero by this operation, it follows from Prof. Sylvester's theory of contravariants that the curve $U=0$ is connected by certain invariant relations with the imaginary circle. I find that U can be

* [Notices and Abstracts...from *Report of the Forty-first Meeting of the British Association for the Advancement of Science*, held at Edinburgh, August, 1871, p. 10. A discussion followed the reading of Prof. Clifford's paper, and a result was that in the same Report, pp. 25, 26, is printed a communication by Sir W. Thomson, *On the General Canonical Form of a Spherical Harmonic of the nth order.* In this Sir W. Thomson answers the question, Can canonical forms not be found in which the nodal conic of each constituent is not resolvable into circular cones and planes?]

expressed in the form

$$U \equiv A^n + B^n + C^n + \ldots$$

where $A = 0$, $B = 0$, ... are great circles touching the imaginary circle, the number of terms being as above. Now if $L = 0$, $M = 0$ be two such great circles meeting in a real point a, and if ϕ be a longitude and θ latitude referred to a as pole, it is easy to see that

$$L^n + M^n = l \sin^n \theta \sin n\phi + m \sin^n \theta \cos n\phi,$$

a sum of two sectorial harmonics, which is the proposed reduction.

When n is less than 5, exceptions of interest occur. For $n = 3$, if we take a, b, corresponding points on the Hessian of the *nodal curve* $U = 0$ (Thomson and Tait, *Treatise on Natural Philosophy*, § 780 [first edition]), and if we call ϕ_1, ϕ_2 the longitudes, θ_1, θ_2 the latitudes referred to these poles, we have

$$U \equiv l \sin^3 \theta_1 \sin 3\phi_1 + m \sin^3 \theta_1 \cos 3\phi_1 + n \sin^3 \theta_2 \sin 3\phi_2 + s \sin^3 \theta_2 \cos 3\phi_2.$$

For $n = 4$, the nodal curve is of the species first noticed by Clebsch, of which many most beautiful properties have been pointed out by Dr. Lüroth. The form U is expressible as the sum of *five* fourth powers; so that if we take a, b real points of intersection of two pairs of them, c a real point on the fifth, calling ϕ_1, ϕ_2, ϕ_3, θ_1, θ_2, θ_3 longitudes and latitudes referred to them, we have

$$U \equiv l \sin^4 \theta_1 \sin 4\phi_1 + m \sin^4 \theta_1 \cos 4\phi_1 + p \sin^4 \theta_2 \sin 4\phi_2 + q \sin^4 \theta_2 \cos 4\phi_2 + r \sin^4 \theta_3 \epsilon^{4\iota\phi_3}.$$

XXVI.

ON THE FREE MOTION UNDER NO FORCES OF A RIGID SYSTEM IN AN N-FOLD HOMALOID. (Provisional Notice.)*

THE problem of the rotation under no forces of a rigid body about a fixed point in ordinary three-dimensional space is the same as the problem of free motion under no forces; for the motion about the centre of inertia takes place as if it were a fixed point. But it is also the same thing as the problem of the free motion of a rigid system on the surface of a sphere, or in elliptic space of two dimensions†. So also the problem of the free motion of a solid in elliptic space of three dimensions is the same as that of the free motion, or motion about a fixed point, in parabolic or homaloidal space of four dimensions. And, in general, the problem of free motion in elliptic space of n dimensions is identical with that of free motion, or motion about a fixed point, in parabolic space of $n + 1$ dimensions.

The form of the problem which is considered in what follows is that which deals with the motion about a fixed point in parabolic space of n dimensions.

* [From the *Proceedings of the London Mathematical Society*, Vol. VII., Nos. 92, 93, pp. 67—70.]

† According to Dr Klein's nomenclature, a space every point of which can be uniquely represented by a set of values of n variables is called elliptic, parabolic, or hyperbolic, when its curvature is uniform and positive, zero, or negative. The geometry of the sphere becomes elliptic when opposite points are regarded as identical.

I.

Let the co-ordinates of a point, referred to a rectangular system, be $x_1, x_2, \ldots x_n$. If this point belongs to a rigid system in motion, its velocity is given by the equations

$$\dot{x}_h = \Sigma p_{hk} x_k \quad (h, k = 1, 2, \ldots n) \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where $p_{hk} = -p_{kh}$, $p_{hh} = 0$. The $\frac{1}{2}n(n-1)$ quantities p are of the nature of rotational velocities of the rigid system. It may be observed that if the vector from the origin to the point x be represented in terms of n unit vectors $\iota_1\ \iota_2 \ldots \iota_n$, satisfying the equations $\iota_h \iota_k = -\iota_k \iota_h$, $\iota_h^2 = -1$, then the velocity of the rigid body may be represented in terms of the $\frac{1}{2}n(n-1)$ products $\iota_h \iota_k$; namely, we may write

$$\rho = \Sigma \iota_h x_h, \quad -2p = \Sigma p_{hk} \iota_h \iota_k,$$

and then the equation (1) may be put into the form

$$\dot{\rho} = Vp\rho.$$

If dm be the element of mass at the point x, its kinetic energy is

$$\tfrac{1}{2}\Sigma \dot{x}^2 dm = \tfrac{1}{2} dm \{\Sigma p_{hk}^2 (x_h^2 + x_k^2) + 2\Sigma p_{hl} p_{kl} x_h x_k\}.$$

Let then $$\int \dot{x}_h^2 dm = \alpha_h, \quad \int x_h x_k \, dm = \beta_{hk},$$

the integrations extending over the whole rigid system; then, if T be the kinetic energy of the system,

$$2T = \Sigma (\alpha_h + \alpha_k) p_{hk}^2 + 2\Sigma \beta_{hk} p_{hl} p_{kl} \ldots\ldots\ldots\ldots\ldots (2).$$

Write now $$q_{hk} = \frac{\delta T}{\delta p_{hk}}, \quad -2q = \Sigma q_{hk} \iota_h \iota_k,$$

then q is the *momentum* of the system; it is a linear function of the velocity, or $q = \phi(p)$, and twice the kinetic energy is the scalar part of the product of velocity and momentum, $2T = Spq$. The equations of motion are $\dot{q} = f$, where f is the system of applied forces, or in the present case of no forces $\dot{q} = 0$; viz., this is equivalent to $\frac{1}{2}n(n-1)$ equations. From this we get the first integrals, $q =$ constant, and (since $0 = Sp\dot{q} = 2\dot{T}$) $T =$ constant.

But these equations are inconvenient, because the α and β are variable, depending upon the position of the body. We must therefore follow Euler in referring the motion to axes moving with the body, and coinciding with the principal axes at the fixed point. This will make all the β vanish; and we shall have $$q_{hk} = (\alpha_h + \alpha_k)\, p_{hk}.$$

From equation (1) we obtain

$$\dot{q}_{hk} = \int dm\,(\ddot{x}_h x_k - x_h \ddot{x}_k) = \Sigma_l \int dm\,\{\dot{x}_l(p_{hl}x_k - p_{kl}x_h) + x_l(\dot{p}_{hl}x_k - \dot{p}_{kl}x_h)\}.$$

Assuming that $\int dm x_h x_k = 0$ when h and k are different, and remembering that $\dot{q}_{hk} = 0$, we may write this equation in the form $$(\alpha_h + \alpha_k)\,\dot{p}_{hk} + (\alpha_h - \alpha_k)\,\Sigma_l p_{hl} p_{lk} = 0 \quad \ldots\ldots\ldots\ldots (3).$$

Here $\dot{p}_{hk}$ means the rate of change of that component of velocity which coincides at the moment with one of the principal components; it must be distinguished from the rate of change of the principal component, which we shall call $(\dot{p})_{hk}$. In general, if the system of axes has the rotational velocities r_{hk}, we shall have $$\dot{p}_{hk} = (\dot{p})_{hk} + \Sigma(p_{hl}r_{kl} - p_{kl}r_{hl});$$

but in the present case the r are equal to the p, because the axes move with the body, and so $\dot{p}_{hk} = (\dot{p})_{hk}$. Thus in the equations (3), which are analogous to Euler's equations for three dimensions, the symbols $\dot{p}_{hk}$ may be understood to mean the rates of change of the principal components of rotation.

If the symbol V_2 is regarded as selecting all the binary products of $\iota_1 \iota_2 \ldots \iota_n$ out of any rational integral function of them, the last equations may be written

$$\dot{q} = V_2 pq;$$

and it may be observed that we have also

$$q = \int dm\, V_2 \rho\dot{\rho}.$$

II.

These equations may be integrated by means of Θ-functions of $n-2, = s$, arguments, one of which is a linear function of the time. It is most convenient to use these in the form employed by Göpel for the case $s = 2$. Let $u_1, u_2, \ldots u_s$ be the arguments, and let $B_{11}, B_{12}, \ldots B_{1s}, \ldots B_{s1}, \ldots B_{ss}$ be s^2 constant quantities, and let

$$U_h = (u_h + 2m_1 B_{1h} + 2m_2 B_{2h} + \ldots + 2m_s B_{sh})^2 = (u_h + 2\Sigma m_k B_{kh})^2;$$

then we shall write

$$G(u_1, u_2, \ldots u_s) = \Sigma_m e^{\Sigma U_h},$$

where the whole numbers m are to take independently all values from $-\infty$ to $+\infty$. It is clear that the function G is unaltered if we simultaneously increase $u_1, u_2, \ldots u_s$ by equimultiples of the quantities $2B_{h1}, 2B_{h2}, \ldots 2B_{hs}$, for this is only increasing m_h by an integer. Moreover, if we determine s^2 quantities A so that

$$4\Sigma_k A_{hk} B_{hk} = \pi i, \quad \Sigma_k A_{hk} B_{lk} = 0,$$

then the function $e^{-\Sigma uu} G$ is unaltered if we simultaneously increase $u_1, u_2, \ldots u_s$ by equimultiples of $4A_{h1}, 4A_{h2}, \ldots 4A_{hs}$. In what follows we shall write for shortness $G(u + X_h)$ instead of $G(u_1 + X_{h1}, u_2 + X_{h2}, \ldots u_n + X_{hn})$, omitting always the last suffix, and mentioning only one argument u.

A linear function of the A, B with coefficients 0 or 1 will be called a *quadrant;* there are clearly 2^{2s} quadrants, if zero be included among them. Let X and Y be two quadrants, A_h the difference between those parts of them which involve the quantities A_{hk}, then the function

$$e^{-\Sigma A u} \frac{G(u + X)}{G(u + Y)} = A l_{X, Y}(u)$$

is $2s$-periodic in the arguments u, the periods being $4A$, $4B$. It is convenient to speak of the *distance* of two quadrants X and Y, meaning the number of coefficients of the A and B which must be changed from 0 to 1, or *vice versâ* to make one of them into the other. This distance may be any of the numbers

1, 2, ... $2s$; and accordingly there are $2s$ really distinct $2s$-periodic functions.

It is, however, possible to form a group of $n, = s+2$, quadrants X_1, X_2, ... X_n having such a relation to the quadrant O that if we write $Al_{hk}(u)$ for $e^{-\Sigma \lambda u} G(u + X_h + X_k) : Gu$, the $\frac{1}{2}n(n-1)$ functions $Al_{hk}(u)$ satisfy the equations

$$\delta_u Al_{hk}(u) = \Sigma c_l Al_{hl}(u) \, . \, Al_{kl}(u) \quad \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

where the δ_u applies to any one of the arguments; but the values of the c will depend upon which argument is taken. In the case of the hyperelliptic functions, the four quadrants X may be taken to be the quantities A_1, A_2, B_1, B_2.

It appears therefore that the equations (3) may be integrated if we write $p_{hk} = \lambda_{hk} Al_{hk}(u)$, where $u_1 = at + e$.

III.

From the $\frac{1}{2}n(n-1)$ equations (3) let us pick out $n-1$, namely

$$-(\alpha_1 + \alpha_h)\dot{p}_{1h} = (\alpha_1 - \alpha_h)\Sigma p_{hk} p_{1k};$$

if we write $(\alpha_1 + \alpha_h) p_{1h} = (\alpha_1 - \alpha_h) \xi_h$, these equations become

$$-\dot{\xi}_h = \Sigma p_{hk} \xi_k.$$

But these are the equations for the velocity of a fixed point ξ relative to the moving axes in $n-1$ dimensions. The rest of the equations (3), if we write in them $p_{1h} = 0$, become the equations for the component rotations in $n-1$ dimensions. Thus the solution for the rotational velocities *and* the position of a point fixed in space, for $n-1$ dimensions, are obtained by diminishing the number of periods in the solution for n dimensions; it consists accordingly of the Al functions expressing the rotation-velocities, and of Rosenhain's combination of Θ-functions and exponentials, expressing the position.

XXVII.

ON THE CANONICAL FORM AND DISSECTION OF A RIEMANN'S SURFACE*.

THE object of this Note is to assist students of the theory of complex functions, by proving the chief propositions about Riemann's surfaces in a concise and elementary manner. To this end I assume only certain results of Puiseux, which are put together at the outset.

I.

Puiseux's Theory of an n-valued Function.

If two variables s and z are connected by an equation of the form $f(s, z), = (s, 1)^n (z, 1)^m, = 0$, each is said to be an algebraic function of the other. Regarding z as a complex quantity $x + iy$, we represent its value by the point whose co-ordinates are x, y, on a certain plane. To every point in this plane belongs one value of z, and consequently, in general, n values of s, which are the roots of the equation $f = 0$. The points of the plane may be divided into those at which the n values of s are distinct, and those at which two or more of them are equal. The latter points are finite in number, and correspond to the roots of the equation which is got by equating to zero the discriminant of f in regard to s. If the roots of this equation are distinct, there are $2(n-1)m$ such points, because the discriminant of the

* [From the *Proceedings of the London Mathematical Society*, Vol. VIII., No. 122, pp. 292—304.]

equation of the nth order in s is of degree $2(n-1)$ in the coefficients, and these coefficients are of the order m in z. But a point at which r values of s become equal corresponds to an $(r-1)$-fold root of the discriminant-equation.

Let us now consider an arbitrary point O of the plane [fig. 31], corresponding to a value z_0 of z, which is not a root of the discriminant-equation. Then the equation $f(s, z_0)=0$ will give n different values for s, which we may call $s_1, s_2 \dots s_n$. If we move along any path from the point O to another point P of the plane, the value of z will change continuously, and each of the quantities $s_1, s_2 \dots s_n$ will also change continuously. If therefore the path OP does not go through a point where two values become equal, these n quantities will be distinct all the way, and each of the n values of s at P will belong to a definite one of the values of s at O. But if the path goes through such a point, two or more of the n quantities will become equal and then diverge again, so that it will be impossible after that to distinguish them so as to say which of these belongs to a particular one of the values at the point O. We cannot always avoid this difficulty by going *round* the point, for it is found that the values at P to which the values at O correspond may depend upon the path OP, so that the correspondence is different for a path which goes to the right of the point and for a path which goes to the left of it. When this is the case, the point is called a branch-point. Suppose that, when we go from O to A, the two values p and q of s at O approach one another and become equal at A; then it is found that the value at P which represents p when we go along the path OBP may represent q when we go along OCP, and *vice versâ*. So that, if we travel along $OBPCO$, round the point A and back to O, the values p and q will change continuously into one another. If more than two values are equal at A, the corresponding values at O may be cyclically interchanged by a path going round A. We shall assume, however, that only two values become equal at each branch-point; and, moreover, that no branch-point is at an infinite distance*.

* Roots of the discriminant-equation which are not branch-points correspond to double points on the curve $f(s, z)=0$. Such points behave, in regard to

A path going along any line from O to very near A, then round A in a very small circle, and then back to O along the same line, will be called a *loop*.

If we start from O and go round any closed curve not including any branch-points, the n values of s at O will be restored in the same order. For the path may be gradually shrunk into a point without crossing any branch-points, so that no two of the n values can become confused at any point of it. The same thing is true if the closed path includes *all* the branch-points. Suppose it a large circle through O; then it may be gradually increased till it coincide with the tangent at O, then curved over on the other side, and shrunk up into a point; and during the whole process the n values will be distinct at every point of the path.

We shall now go on to shew that this n-valued function, which we have spread out upon a single plane, may be represented as a *one*-valued function on a surface consisting of n infinite plane sheets, supposed to lie indefinitely near together, and to cross into one another along certain lines. This surface is called a RIEMANN'S surface; we shall demonstrate its existence at the same time that we shew how to construct it in the most convenient form.

II.

Construction of the Riemann's Surface.—Lüroth's Theorem.

Draw loops from O [fig. 32] to all the branch-points, and let the first, A, interchange the values p and q. If we go round all the loops successively, starting with the value p at O, we must, as we have seen, come back to that value; but this may happen before we have used all the loops. Let B be the first branch-point after going round which the value p is restored. Draw a line from A to B cutting all the loops which alter p, but none of the others. Then, if we go round any of the

the function s, like two coincident branch-points belonging to the same pair of values, and they have no influence on the connection of the different values of s.

branch-points between A and B without crossing the line AB or going round any other branch-points, we shall not alter the value p.

Suppose that A interchanges pq, B interchanges ps, and that the branch-points between A and B are 1, 2, 3, 4, interchanging respectively qr, rs, hk, pl. The value q must in fact be changed into the value p through a longer or shorter series of values; the loops interchanging hk and pl are put in as examples. Now if we go round 4 by the dotted loop passing round outside A, the effect is the same as going in succession round A, 1, 2, 3, 4, 3, 2, 1, A. By the time we have gone round A, 1, 2, 3, we cannot have the value p, for that is first restored by B; and we cannot have the value l, for then 4 would restore the value p. Hence we have some value which is not altered by the loop to 4; and consequently, when we retrace our path, we shall come back to the value p.

Next, let us draw a loop to B which passes within the line AB, but goes round all the included branch-points, as in the figure. The effect of this loop will be to change q into p; for it is the same thing as going round 1, 2, 3, B, 3, 2, 1. Now the effect of 1, 2, 3, B is to change q into p, and this p is not altered in coming back because all the branch-points which alter p are outside the line AB.

Suppose then that all the branch-points of this group which alter p are connected with O by loops going round A, so that they no longer alter p; and that B is connected with O by the loop just described, so that no branch-points are contained in the triangle AOB.

Starting now from this new loop OB, with the value p, let us go round all the loops as before from left to right. We know that when all the loops have been gone round, ending with OA, the value p must be restored. If it is not restored before we have gone round OA, we must draw a line BA cutting all the loops which change the value p but none of the others. But if the value p is restored before we have gone round OA, say after going round OC; then we must draw a new loop to C, going round all the branch-points between A and C except those which change the value p. This new loop will, by our previous

reasoning, change p into q. Hence, if the value p is restored before we have gone round OA, we can make a new loop OC which changes p into q; and this comes next to OB. To those branch-points whose loops have been cut by this new loop we must draw new loops going round to the right of C, so as not to cut OC. The figure comes then into this form [fig. 33], containing

(1) Loops to the left of OA which do not change the value of p, like the dotted loop $O4$ in the previous figure;

(2) Three consecutive loops OA, OB, OC which change p into q;

(3) Loops to the right of OC which may or may not change p.

If now we start with the loop OC and proceed to the right, the value p must be restored *before* we have gone round OA; for, starting with OA and going all round, we must restore the value p in the end. Let p then be restored by OD; and draw a line CD cutting all those loops which change p, but none of the others. Replace the loops which change p by new ones going round between B and C; and replace OD by a new loop going outside all the branch-points whose loops do not alter p. The figure now consists of these elements:

(1) Two triangles AOB, COD, containing no branch-points, and such that the loops OA, OB, OC, OD interchange p and q;

(2) Loops between OB and OC which do not change p;

(3) Unknown loops between OD and OA.

About these unknown loops we may make three suppositions.

First, suppose that none of them change p. Then the value p cannot be altered by any closed curve starting from O and returning to it which does not cut either of the lines AB, CD.

Secondly, suppose that some of these loops change p, but that, when we start with the loop OD and go round to the right, the value p is first restored by OA or OC. (It is clear that it cannot be first restored by OB, because the two loops OA, OB, taken together, make no change in any value; nor by any loop

between OB and OC, for none of them change p.) Then we must join D with A by a line cutting all the loops which change p, but no others; and B with C by a line cutting none of the loops between OB and OC. In that case the value p cannot be altered by any closed curve starting from O and returning to it which does not cut either of the lines BC, DA.

Thirdly, suppose that the value p is restored *before* we come to OA, say at OE. Then we must proceed as before, finding a new line EF which shall have the properties of AB or CD. The figure will then consist of three triangles AOB, COD, EOF, containing no branch-points, and such that the loops OA, OB, OC, OD, OE, OF interchange p and q; loops between OB and OC, and between OD and OE, which do not change p; and unknown loops between OF and OA.

It is clear that this process must ultimately stop, and then we shall be left with a finite number of lines such that, if we start from O, follow any continuous path, and come back again, without crossing any of these lines, we shall not alter the value p. The lines are either AB, CD, EF, &c., or else they are BC, DE, &c.; in either case the loops $OA, OB, \ldots$ interchange p and q.

It follows that, if we take an infinite plane sheet and cut it through along these lines, we may consider a single value of the function s to be attached to every point of the sheet in such a way that this value varies continuously when we move about continuously in the sheet; but there will be different values on the two sides of any cut—namely, we must attach to every point P of the sheet that value of s which changes continuously into p when we go from P to O without crossing any of the cuts. There is only one such value; for if two different paths from O to P gave different values at P, it would be possible to change the value p by means of a closed curve returning to O; and this we have proved not to be the case.

When the lines cut through are AB, CD, ..., the triangles AOB, COD, ... contain no branch-points; but when the lines are BC, DE, ..., the triangles BOC, DOE do in general contain branch-points. We may, however, draw new loops to C, E, ... so as to exclude these branch-points, and the new loops will still change p into q. For no closed curve going round B and C

so as not to cut BC can change the value p, by what we have already proved; but the loop OB changes p into q, therefore OC must change q into p.

We shall assume then that the cuts are AB, CD, ..., and that the triangles AOB, COD, ... contain no branch-points.

Now let us deal with the value q at O in the same way as we have dealt with the value p. It is first to be observed that a path going round one or more of the lines AB makes no change in *any* value at O; so that, if we agree never to cross these lines, we may leave the branch-points A, B, ... entirely out of consideration.

This being so, let us take a loop which changes q into some other value, say r. There must be such a loop, if the function is more than two-valued; for otherwise the values p, q would form a two-valued algebraic function of z, and the expression $f(s, z)$ would have a factor of the second degree in s.

Starting then with this loop, we may proceed in exactly the same way as before, and draw lines $A'B'$, $C'D'$, ... such that a closed curve, starting from O and coming back to it without cutting any of these lines or any of the previously drawn lines, will not alter the value q. Moreover, we shall have drawn loops OA', OB', ..., each of which changes q into r, and such that the triangles $A'OB'$, $C'OD'$, ... contain no branch-points. And since our previous triangles AOB, COD, ... contained no branch-points, it will not have been necessary to cut through them in drawing the new lines $A'B'$, $C'D'$, ...

We shall now speak of the first set of lines AB, CD,... as the lines (pq), and of the second set as the lines (qr).

Let us take two infinite plane sheets, cut them both through along the lines (pq), but only the second one along the lines (qr). To every point of the *first* sheet we will suppose attached that value of s which is arrived at by continuous change of the value p at O; and to every point of the *second*, that value which is arrived at by continuous change of the value q at O.

In each sheet there will be a finite difference in the values on the two sides of each of the cuts (pq); but the value on one side in the upper sheet will be equal to the value on the other side in the lower sheet. At the cut AB, for example, the value

continuous with p on the side next to O is equal to the value continuous with q on the side remote from O; because a path taken round A or B from O and back again changes the value p continuously into the value q.

Thus, if we take p, q to denote values at the cut continuous with p, q at O, they will be situated as in the figure [fig. 34], which represents a section across AB perpendicular to the two sheets. If then we make the two sheets cross one another along the lines p, q, as here represented [fig. 35], then these two values will be continuously distributed on the double-sheeted surface so formed.

We may now continue the process with the value r. We must first find a loop which changes r into some other value, say t, and then proceed as before, taking care not to cross the lines qr. (We may cross the lines pq as often as we please, provided that we have not previously crossed the lines qr; for these lines can have no effect upon r unless it has been previously changed into q.) Thus we shall draw lines rt such that the value r cannot be altered by a closed curve not cutting the lines qr or rt, and having their extremities joined to O by loops which change r into t. If we take, then, a third sheet, cut it through along the lines qr and rt, and then join it crosswise to our second sheet along the lines qr; the three values pqr may be continuously distributed on this three-sheeted surface.

By proceeding in this way it is clear that we shall construct an n-sheeted surface, the sheets of which are connected chainwise by cross lines, so that the first is connected only with the second, the second with the third, and so on; but there is no direct connection except between consecutive sheets. And the n values of the function may now be attached to the points of this surface, so that one value only belongs to each point, and that in moving this point about on the surface the value belonging to it always changes continuously. Thus, if we start from a given point of the surface (on a given sheet), and travel by any path so as to come back to the same point (on the same sheet), we shall in all cases return to the former value of the function s.

The theorem that the Riemann's surface may be so con-

structed that the sheets are only connected *chainwise—i. e.*, so that there are no cross-lines except between consecutive sheets —is due to Dr. Lüroth.

III.

Clebsch's Theorem.

All the links between successive sheets except the last may be made to consist of one cross-line only.

First, we shall prove that, if there are two or more lines (pq), one of them may be converted into a line (qr).

The original position of the two lines (pq) and the line (qr) is drawn in fig. [36]. If we move the line qr, keeping, of course, its ends fixed, the effect is to interchange the sheets QR in the area over which it moves; so that, by passing it over the line (pq) on the right, we change this into a line (pr). The position is then as in fig. [37]. If now we pass the remaining line (pq) over this line (pr), we change it into a line (qr); thus we are left with two lines (qr) and one line (pq). [Fig. 38.]

In this way we may convert all but one of the lines (pq) into lines (qr). Then we may convert all but one of the lines (qr) into lines (rs); and so on. Then the first $n-1$ sheets will be connected chainwise by one cross-line each, and the last two by all the remaining cross-lines.

The Riemann's surface is now said to be in its canonical form.

The process of transformation may be made clearer by looking at a section of the three sheets by a plane perpendicular to them cutting the lines pq, qr, pq [figs. 39, 40, 41].

IV.

Transformation of the Riemann's Surface.

The Riemann's surface now consists of n infinite plane sheets, such that the sheet 1 is connected with 2 by a single cross-line, 2 with 3 by another cross-line, and so on; but $(n-1)$

with (n) by a number of cross-lines which we shall call $p + 1$. Thus the whole number of cross-lines is $n - 2 + p + 1 = n + p - 1$. If w is the number of branch-points, this is twice the number of cross-lines, or $w = 2(n + p - 1)$. Hence $p = \frac{1}{2} w - n + 1$.

Let now this n-fold plane be inverted in regard to any point outside it, so that it becomes an n-fold sphere passing through the point. Any two successive sheets of the sphere will be connected by one cross-line, except the two outside sheets, which are connected by $p + 1$ cross-lines.

To every point of this n-sheeted spherical surface will correspond one value of the function s, namely, that which belongs to the corresponding point upon the n-fold plane. As for the centre of inversion, it is to be regarded as n distinct points upon the several sheets, corresponding to the n values of s when $z = \infty$.

We shall now prove that this n-fold spherical surface can be transformed without tearing into the surface of a body with p holes in it.

First, suppose we have only two sheets, connected by a single cross-line which joins the branch-points AB. Let the figure [42] represent a section by the plane which bisects AB at right angles.

Suppose each hemisphere of the inner sheet to be moved across the plane of the great circle containing AB (indicated by the dotted line in the figure), so that the points m, n change places. In this process the two hemispheres will have to penetrate and cross each other; but this may be supposed to take place without altering the continuity of either. Each point may be supposed to move on a straight line perpendicular to the dotted plane, till it coincides with what was its reflexion in regard to that plane. The effect on the cross-line will be to change it from the form drawn in fig. [42] to that drawn in fig. [43]; instead of the two sheets crossing along the line, each of them will be doubled under it. The result is that, if we now look down on the double sphere from a point vertically over the line AB, we shall see a spherical shell with a hole in it, in the form of a slit along the line AB [fig. 44]. Conceive the spherical shell to be made of india-rubber or some more elastic substance ;

then by mere stretching, without tearing, the slit may be opened out until the shell takes the form of a flat plate; that is, of a body with *no* holes in it.

Next, consider a two-sheeted spherical surface with $p + 1$ cross-lines, and suppose them all arranged along the same great circle; which may obviously be done by stretching, without tearing, the surface. Let this great circle be the one represented by the dotted line in figs. [42] and [43]. Then we may apply to the inner sheet the same process as before; viz., we may interchange the two hemispheres into which the sheet is divided by the dotted plane. The effect is to convert all the cross-lines into slits or holes in a spherical shell; and we have supposed that there are $p + 1$ of these. One of the slits may be stretched out in the same way as AB was before, so as to convert the spherical shell into a flat plate; but in this flat plate there will remain p holes. A double sphere with $p + 1$ crossing lines is thus converted, without tearing, into the surface of a body with p holes in it.

Lastly, suppose that the inner sheet of this two-sheeted sphere is connected by one cross-line with a third inside sheet, the third sheet by one cross-line with a fourth inside it, and so on, until there are n sheets. Let the inner sheet of all be reflected in regard to the plane of the great circle through its crossing line, so that it makes with the sheet next to it a spherical shell with one hole in it. Then, without tearing, the inner sheet may be shrunk up until it merely covers over this hole. The same process may now be applied to shrink up the second sheet into the third, and so on, until we are left with only the two outside sheets connected by $p + 1$ cross-lines. These, however, as we have seen, may be converted, without tearing, into the surface of a body with p holes in it. Hence the proposition follows, that *an n-sheeted Riemann's surface with w branch-points may be transformed, without tearing, into the surface of a body with* $p, = \frac{1}{2}w - n + 1$, *holes in it.*

V.

The Number of Irreducible Circuits.

A closed curve drawn on a surface is called a *circuit.* If it is possible to move a circuit continuously on the surface until it shrinks up into a point, the circuit is called *reducible;* otherwise it is *irreducible.* In general there is a finite number of irreducible circuits on a closed surface which are *independent,* that is, no one of which can be made by continuous motion to coincide with a path made out of the others. All other irreducible circuits can then be expressed as compounds of these independent ones. For example, on the surface of a ring (*i.e.*, of a body with one hole through it) there are two independent irreducible circuits; one *round the hole,* as *abc* [fig. 45], and one *through the hole,* as *ade.* If a circuit goes neither round the hole nor through the hole, it can be shrunk up into a point. If it cannot be so shrunk up, it must go a certain number of times round or through the hole or both, that is, it may be made up of circuits like *abc* and *ade.*

In the same way we may see that, on the surface of a body having p holes through it, there are $2p$ independent irreducible circuits; one *round* each hole, and one *through* each hole. For simplicity consider the case $p = 3$. We suppose the body in the form to which we reduced the Riemann's surface, namely, that of a flat plate, represented by figs. [46] and [47], in which A, B, C are the holes. The circuits through each hole are so drawn as to connect the hole directly with the outer rim, like the circuit which is drawn through the hole A. A circuit passing through *two* holes, as B, C [fig. 46], may be moved continuously till it consists of two circuits going through the two holes separately. Similarly, a circuit round two or more holes, as B, C [fig. 47], may be pinched at various points until it is made up of circuits round the separate holes. Such a circuit as *abcd* [fig. 46] may be moved into the form *abcd* [fig. 47], in which it consists of two circuits going through the hole A, but in opposite directions. On this account it may be called a *nugatory* circuit.

VI.

The Canonical Dissection.

Suppose now that it is desired to cut through the Riemann's surface in such a way that it shall still hang together, but that it shall no longer be possible to draw an irreducible circuit upon it. This we may do if we successively prevent the different kinds of irreducible circuits considered in the last section. To prevent the possibility of going *round* any hole, we must cut the surface along a circuit which goes *through* the hole. To prevent the passage *through* a hole, we must cut through a circuit which goes *round* a hole.

Let us make sections a_1, a_2, a_3 [figs. 48, 49] round the holes, and b_1, b_2, b_3 through the holes. Then we shall have prevented the drawing of any irreducible circuits except nugatory ones, like *abcd* in the previous figures. To prevent these also, we may cut the surface along the line c_1 which goes from p to q, that is, from a point on b_2 to a point on b_3, and along the line c_2 which goes from q to r, that is, from a point on b_3 to a point on b_1. We must not cut from r to p also, for then we should divide the surface into two separate parts. We may now open out the upper and under portions of the surface in fig. [48], until it assumes the form of fig. [49]. It then becomes obvious that all our cuts form a continuous line, which is now the boundary of the surface, and is made up of the pieces (beginning at p and going round to the right) $c_1 b_3 a_3 b_3 c_2 b_1 a_1 b_1 a_1 b_1 c_2 b_3 a_3 b_3 c_1$. Moreover, it is a matter of intuition that no irreducible contour can now be drawn on the surface.

This system of cuts is called a *canonical dissection* of the surface. In the general case it consists of p cuts a going round the holes, p cuts b going through them, and $p-1$ cuts c joining b_2 to b_3, b_3 to b_4, ... b_p to b_1, but not b_1 to b_2. The cuts c may, if we like, join the a-cuts together, or generally they may join the systems (ab) together, a *system* meaning an a-cut and a b-cut belonging to the same hole. In fact, the c-cuts are only of importance as completing the single boundary of the surface,

and so enabling us to see that no irreducible circuit is any longer possible.

It only remains to translate this result so that it may be applicable to the original form of the Riemann's surface, viz., an n-fold plane. We shall do this in the case $p = 2$, which will sufficiently explain the general case. We have now two sheets connected by three cross-lines mn, pq, rs [fig. 50]. One of these must be chosen to represent the outer rim of our flat plate; the other two will then correspond to the holes in it. Let mn, pq represent the holes, and rs the outer rim; lines in the upper sheet shall be drawn in full, and lines in the lower sheet shall be dotted. Then we must first make cuts a_1, a_2, which go round the holes mn, pq; these may lie entirely in the upper sheet. Next we must make cuts b_1, b_2, which connect the holes respectively with the outer rim rs. These cuts lie partly in the upper sheet, where they intersect the cuts a, and partly in the under sheet. Lastly, we must connect the system $a_1 b_1$ with the system $a_2 b_2$ by the cut c; this is drawn in the figure from b_1 to b_2 in the under sheet. It is impossible to draw an irreducible circuit on the two-fold plane when it is thus dissected*.

In general, we have proved that in the n-sheeted Riemann's surface which represents the function s determined by the equation $f(s, z) = 0$, there are $p + 1$ cross-lines such that if one be taken to represent the rim, and the rest holes, of a flat plate, the surface may be dissected into one on which no irreducible contour is possible by the following process:—Cut the surface along curves a each of which goes round one of the cross-lines taken to represent holes, on one of the sheets of the surface which cross at that line. Connect each of these lines with the one taken to represent the rim by a cut b along a closed curve which crosses each of the two cross-lines once. Then connect the systems (ab) chainwise by $p - 1$ cuts c.

* It is to be understood that a circuit is *reducible* when all parts of it can be continuously moved away to infinity without crossing any branch-point; because in this theory infinity counts as a single point.

Fig. 31.

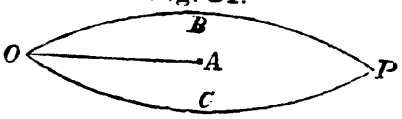


Fig. 32.

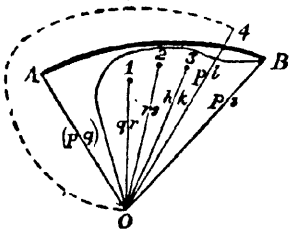


Fig. 33.

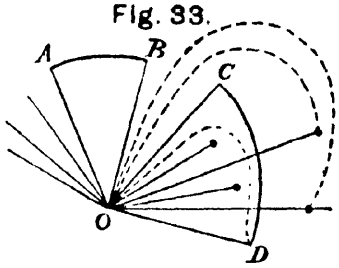


Fig. 34.

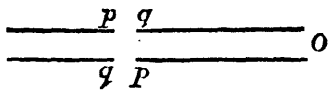


Fig. 35.

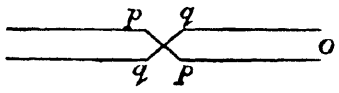


Fig. 36.

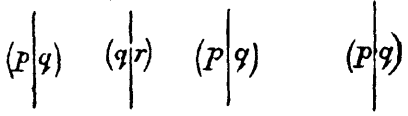


Fig. 37.

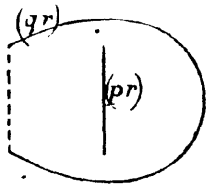


Fig. 38.

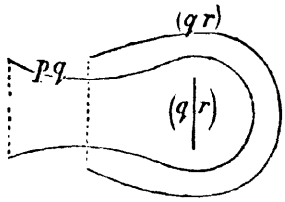


Fig. 39.

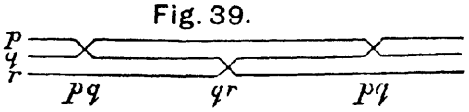


Fig. 41.

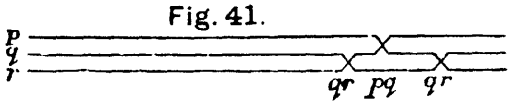


Fig. 40.

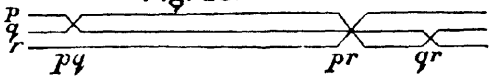


Fig. 42.

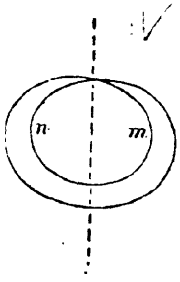


Fig. 43.

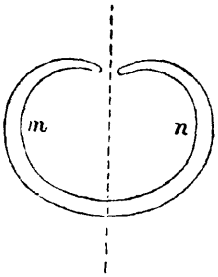


Fig. 44.

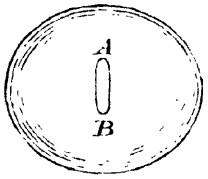


Fig. 45.

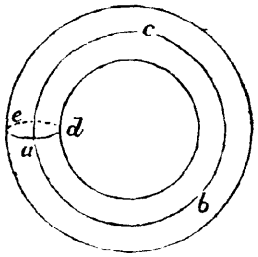


Fig. 46.

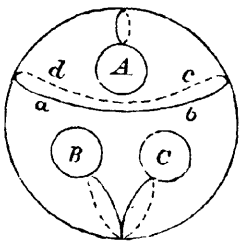


Fig. 47.

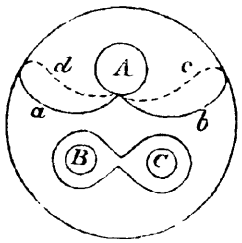


Fig. 48.

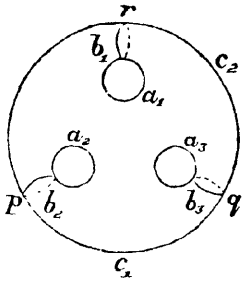


Fig. 49.

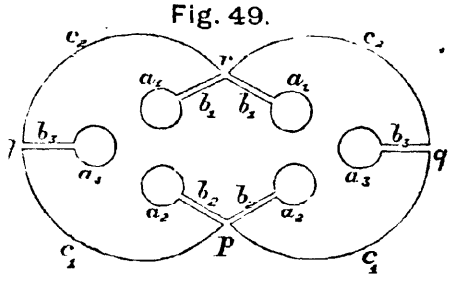
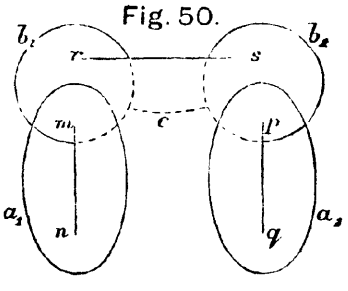


Fig. 50.



face p. 254.

XXVIII.

REMARKS ON THE CHEMICO-ALGEBRAICAL THEORY.

(Extract from a letter to Mr Sylvester*.)

"THE new Journal [see foot-note] I look forward to with the greatest interest: it will be the only English periodical in which one will have room to print formulæ, except the *Philosophical Transactions*. I had designed for you a series of papers on the application of Grassmann's methods, but there is only one of them fit for printing yet†. It is an *explanation* of the laws of quaternions and of my biquaternions by resolving the units into factors having simpler laws of multiplication; a determination of the compounding systems for space of any number of dimensions; and a proof that the resulting algebra is a *compound* (in Peirce's sense) of quaternion algebras. It thus appears that quaternions are the last word of geometry in regard to complex algebras. Another of them was to be about the very thing you speak of, which was communicated to the British Association at Bristol, *not* Bradford. There is no question of reclamation, because the whole thing is really no more than a translation into other language of your own theories published years ago in the *Cambridge Mathematical Journal*. I have a strong impression that you will find there the analogy of covariants and invariants to compound radicals and saturated molecules.

I consider forms which are linear in a certain number of sets of k variables each. To fix the ideas, suppose $k=2$ and that I have altogether 6 sets of 2 variables each, namely

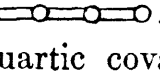
$$x_1x_2,\ y_1y_2,\ z_1z_2,\ u_1u_2,\ v_1v_2,\ w_1w_2.$$

* [From the *American Journal of Mathematics, Pure and Applied*, Vol. I. No. 2, pp. 126—128.]

† [This is xxx. of the present volume.]

Suppose the forms are

$$(xyzu),\ (yzvw),\ (xv),\ (uw)\ ;$$

viz. $(xyzu)$ means an expression separately linear and homogeneous in the x, the y, the z, and the u, and so for the rest. I observe that in these four forms each set of variables occurs twice. This being so, there is one invariant of the four forms, which is invariant in regard to *independent* transformations of the six sets of variables. This you knew thirty years ago. All I add is: *to obtain this invariant,* regard the *variables as alternate numbers*, and *simply multiply all the forms together.* By *alternate* numbers I mean those whose multiplication is polar ($xy = -yx$) and whose squares are zero. The product of the forms will then be equal to the invariant in question multiplied by the product of all the variables. The quartic forms may be represented by the symbol -◇-, the quadratics by -o-. Thus the invariant $(xyzu)\ (yzvw)\ (xv)\ (uw)$ will be represented by the figure ; whereas, $(xyzu)\ (yzvw)\ (xu)\ (vw)$ is this form  . The former is clearly the product of the two quartic covariants got by cutting it across the dotted lines; while the latter is the product of the quadri-covariants , . A *bond* between two forms means a set of variables common to them. Of course, we may regard two or more of the forms as identical and so form invariants of a single form; thus is the discriminant of a cubic*...Of course, the main thing is to pass from this system of separate variables to that in which the same variables occur to higher orders in the same form, or back again—what you call 'unravelment'....

The part of the theory which astonished me most is its application to *intergredient* variables when the number in a set is greater than 3,—such as the six co-ordinates of a line in the case of quaternary forms. When the original variables are regarded as alternate numbers, these intergredients are simply their binary products. Thus by simply multiplying the linear forms representing two planes, we get one intergredient form representing their line of intersection. And so generally, whatever be the number of variables in a set, the intergredient variables are merely their products so many together. With

this understanding, the product of a set of forms in which the variables are regarded as alternate numbers is the *only* invariant or covariant of the forms which possess certain definite characters of invariance.

The ordinary theory of symmetrical forms seems to me to bear the same relation to this one (of forms linear in several sets of variables) that a boulder does to a crystal—all the angles rounded off so that you can't see through it so clearly...." †

† [Dr Sylvester has appended several interesting notes, from which a few extracts are given here. "I think Prof. Clifford overstates the obligations which he alleges to my previous papers. At all events he has more than reconquered his title to the merit of the first conception by the completeness he has imparted to it...In a word he has found the universal pass-key to the *quantification of the graphs*...'All that Prof. Clifford adds' is the very pith and marrow of the matter which before was wanting." Dr Sylvester further remarks, "I will take the example of this figure [cf. * in text] to illustrate Prof. Clifford's rule for finding the *algebraical content* of the graph. Let the bonds be called $\begin{matrix} x \\ y \end{matrix}\ z\ \begin{matrix} t \\ u \end{matrix} v$. Then there will be four forms corresponding to the four apices or atoms, viz.

$$(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)\ (x_1, x_2)\ (y_1, y_2)\ (z_1, z_2),$$
$$(b_1, b_2, \ldots\ldots\ldots\ldots b_8)\ (z_1, z_2)\ (t_1, t_2)\ (u_1, u_2),$$
$$(c_1, \ldots\ldots\ldots\ldots c_8)\ (t_1, t_2)\ (u_1, u_2)\ (v_1, v_2),$$
$$(d_1, \ldots\ldots\ldots\ldots d_8)\ (v_1, v_2)\ (x_1, x_2)\ (y_1, y_2),$$

where all the x, y, z, t, u, v letters are to be regarded as *polar elements*. [Dr Sylvester objects to the term *alternate numbers* in this connexion.] Take the polar product of these forms; the coefficient of

$$x_1 \cdot x_2 \cdot y_1 \cdot y_2 \cdot z_1 \cdot z_2 \cdot t_1 \cdot t_2 \cdot u_1 \cdot u_2 \cdot v_1 \cdot v_2$$

will be an invariant of three lineo-lineo-linear forms.

If we make the values identical for the same index, whatever the letter which it affects, it becomes an invariant of a single lineo-lineo-linear form; and finally if we make the coefficients of $x_1y_1z_2$, $y_1z_1x_2$, $z_1x_1y_2$ all alike, and again the coefficients of $x_2y_2z_1$, $y_2z_2x_1$, $z_2x_2y_1$ all alike, and identify the letters x, y, z, the form becomes a binary cubic and the invariant becomes its discriminant. We know *à priori* by my permutation-sum test that the algebraical content above indicated will not vanish because

$$\Sigma\,(a-b)^2\,(a-d)^2\,(a-c)\,(b-d)$$

is not zero, whereas the algebraical content of the figure formed by turning round one of each pair of the doubled lines into the position of the two diagonals respectively *will* vanish because the permutation-sum of

$$\Sigma\,(a-b)\,(b-c)\,(c-d)\,(d-a)\,(a-c)\,(b-d)$$

is zero."]

*XXIX.

NOTES ON QUANTICS OF ALTERNATE NUMBERS, USED AS A MEANS FOR DETERMINING THE INVARIANTS AND COVARIANTS OF QUANTICS IN GENERAL*.

THE term *alternate numbers* means a set (λ_1, λ_2) or sets of numbers which satisfy the following relations:

$$\lambda_1^2 = 0, \quad \lambda_2^2 = 0, \quad \lambda_1\lambda_2 + \lambda_2\lambda_1 = 0 \ldots\ldots\ldots\ldots (1);$$

to which it is usual to add

$$\lambda_1\lambda_2 = 1.$$

The above set is binary, but there may be ternary, &c. sets, or sets consisting of any number of letters $\lambda_1, \lambda_2, \ldots$

By a *Quadratic Form* is here meant an expression—lineo-linear in two sets of such numbers regarded as variables, say λ_1, λ_2; μ_1, μ_2, such as

$$a_{11}\lambda_1\mu_1 + a_{12}\lambda_1\mu_2 + a_{21}\lambda_2\mu_1 + a_{22}\lambda_2\mu_2 \ldots\ldots\ldots\ldots (2).$$

This may be also denoted, for shortness, by the symbol $a \mid 12$, or even by 12, where the 1, 2 refer to the two sets of variables λ, μ.

* [From the *Proceedings of the London Mathematical Society*, Vol. x. No. 148, pp. 124—129. "This is the substance of some fragments found amongst the papers of the late Professor Clifford. The only published explanation of the method with which I am acquainted, is contained in a letter to Professor Sylvester (see XXVIII.): 'I consider forms which are linear in a certain number of sets of k variables each....The product of the forms will then be equal to the invariant in question multiplied by the product of all the variables.'" SP.]

In this case there is one invariant only which allows of the variables being separately transformed, namely, the discriminant, which is got by squaring the form. We have, in fact, by the properties of alternate numbers,

$$(a \mid 12)^2 = -2(a_{11}a_{22} - a_{12}a_{21}) = -2D, \text{ suppose}.....(3).$$

But if the variables are transformed by the same substitution, there is a universal covariant, $\lambda_1\mu_2 - \lambda_2\mu_1$, which may be denoted by (12), or by (21); for $\lambda_1\mu_2 - \lambda_2\mu_1 = \mu_1\lambda_2 - \mu_2\lambda_1$, by the property of alternate numbers.

If we replace the μ's by x, an ordinary, not an alternate, number, we get a linear function of the λ's, whose square vanishes in virtue of the relations (1). Thus we have $(a \mid 1x)^2 = 0$, and consequently such a product as $(a \mid 1x)(a \mid 1y)$ must be divisible by $(xy) = x_1y_2 - x_2y_1$ since it vanishes when x and y represent the same point; in fact, if we write the expressions in full, thus,

$$(a \mid 1x) = (a_{11}x_1 + a_{12}x_2)\lambda_1 + (a_{21}x_1 + a_{22}x_2)\lambda_2,$$
$$(a \mid 1y) = (a_{11}y_1 + a_{12}y_2)\lambda_1 + (a_{21}y_1 + a_{22}y_2)\lambda_2,$$

actual multiplication gives

$$(a \mid 1x)(a \mid 1y) = a_{11}x_1 + a_{12}x_2,\quad a_{21}x_1 + a_{22}x_2 = a_{11},\ a_{12} \times x_1,\ x_2$$
$$a_{11}y_1 + a_{12}y_2,\quad a_{21}y_1 + a_{22}y_2,\quad a_{21},\ a_{22}\quad y_1,\ y_2.$$

But $a_{11}a_{22} - a_{12}a_{21}$ may be regarded as the discriminant of any quadratic form having a's for its coefficients, say any form $(a \mid 18)$. And, since it was shown, by equation (3), that the square of such a form is equal to minus twice its discriminant, it follows that the above equation may be written thus,

$$-2(a \mid 1x)(a \mid 1y) = (a \mid 18)^2(xy) .....(4).$$

Note.—The transformation

$$y_1, y_2 = \begin{vmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{vmatrix} (x_1, x_2)$$

may also be written

$$a_{11}x_1y_1 + a_{21}x_1y_2 + a_{12}x_2y_1 + a_{22}x_2y_2 = 0 = (a \mid yx),$$

where the y's are arbitrary functions satisfying identically

$$y_1y_1 + y_2y_2 = 0.$$

In like manner, a second transformation

$$z_1, z_2 = \begin{vmatrix} b_{11}, & b_{12} \\ b_{21}, & b_{22} \end{vmatrix} (y_1, y_2)$$

may be represented by $(b \mid zy) = 0$; and the result of the two by

$$(b \mid zy)\,(a \mid yx) = 0.$$

There are one or two other formulæ which it will be convenient to notice before proceeding further. If we multiply the form $(a \mid 12)$ by (12), we get

$$(12)\,(a \mid 12) = a_{21} - a_{12} = s \text{ (suppose)} \ldots\ldots\ldots\ldots (5).$$

Also, if we suppose the coefficients to remain unaltered, viz., if $(a \mid 21)$ represents $a_{11}\mu_1\lambda_1 + a_{12}\mu_1\lambda_2 + \ldots$, then

$$(a \mid 12) + (a \mid 21) = -s\,(12) \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

or

$$2\,(a \mid 12) = (a \mid 12) - (a \mid 21) - s\,(12),$$

the quantity s vanishing when the form is symmetrical, i.e. when $a_{12} = a_{21}$. Again, if the coefficients be supposed to remain unaltered, so that

$$(a \mid 13) = a_{11}\lambda_1\nu_1 + a_{12}\lambda_1\nu_2 + \ldots,$$

then it will be found that

$$\left.\begin{aligned} (12)\,(a \mid 23) &= -(a \mid 13) \\ (13)\,(a \mid 12) &= -(a \mid 32) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Such multiplication is in fact tantamount to a substitution of variables, from 2 to 1 (i.e. from μ to λ), or from 1 to 3 (i.e. from λ to ν). This theorem, as will readily be seen, is not restricted to quadric forms; but if a, b be any two different forms, and if 1 be a set of variables in a and not in b, and 2 a set of variables in b and not in a, then, making the same supposition as in b, viz., that the constants a, b remain the same on both sides of the equation, we shall find that

$$(12)\,(a \mid 1\ldots)\,(b \mid 2\ldots) = (a \mid 2\ldots)\,(b \mid 2\ldots) = (a \mid 1\ldots)\,(b \mid 1\ldots).$$

With reference to the universal covariant $(12) = \lambda_1\mu_2 - \lambda_2\mu_1$, it may be remarked that we may consider such covariants for

more than two sets of variables; and we shall then obtain the following formulæ*:

$$(12)(23) = -(13), \quad (12)(23)(34) = (14), \ldots \quad \ldots\ldots(8),$$

of which the following is a consequence,

$$(12)(23)(31) = 0 \ldots\ldots\ldots\ldots\ldots\ldots(9).$$

It is clear that quadratic forms can only be combined in a *chain*†, which, when open, gives a quadratic covariant; when closed, an invariant. We can now show that a closed chain of $2n$ sides is equal to $\pm 2D^n$, while a chain of an odd number of sides vanishes.

Take a chain of four sides, (12), (23), (34), (41); then, since

$$(12)(23)(34)(41) = (12)(23) \times (34)(41);$$

and since, by (3) and (4),

$$2(12)(23) = (a \mid 28)^2 (31) = -2D(31),$$
$$2(34)(41) = (a \mid 48)^2 (13) = -2D(13),$$
$$(31)(13) = 2,$$

it follows that

$$(12)(23)(34)(41) = 2D^2 \ldots\ldots\ldots\ldots (10).$$

In like manner any chain of an even number of sides may be resolved into a power of the discriminant multiplied by a chain of determinants of the alternate variables. The product of these latter is ± 2, according as the number of terms is even or odd. Hence, generally, a chain of $2n$ sides is $2(-D)^n$.

* [This and other parts of the present paper may be compared with Spottiswoode "On Determinants of Alternate Numbers," *Proceedings of the London Mathematical Society*, Vol. VII., p. 100. SP.]

† [I have, in this paragraph, retained the language of the original MS., although the term *chain* is not here explained. The author appears to have had in his mind a theory which he propounded verbally at the meeting of the British Association at Bristol, and to which, in his latter days, he attached great importance and devoted much time. Some indications of it will be found in the *American Journal of Pure and Applied Mathematics*, Vol. I. p. 127 [cf. XXVIII.]. Several notes relating to the subject have been found amongst his papers, but as they are almost exclusively memoranda without explanation, it is still uncertain whether they can be published. SP.]

A chain of $2n+1$ sides may, by analogous processes, be reduced to the product of a determinant of alternate variables by a form containing those variables, and must therefore vanish for symmetrical forms. For unsymmetrical it contains s as a factor. In this case the fundamental formula must be modified as follows:

$$(a \mid x1)\,(a \mid 1y) = (a \mid xy)\,s - (xy)\,D;$$

or, with alternate numbers in the place of x, y,

$$(a \mid 12)\,(a \mid 23) = (a \mid 13)\,s - (13)\,D.$$

Multiplying now into $(a \mid 31)$, we get

$$(a \mid 12)\,(a \mid 23)\,(a \mid 31) = (a \mid 13)\,(a \mid 31)\,s + sD$$

in virtue of equation (5).

But $$(a \mid 13)\,(a \mid 31) = (a_{11}\lambda_1\mu_1 + \ldots)\,(a_{11}\mu_1\lambda_1 + \ldots)$$
$$= (2a_{11}a_{22} - a^2{}_{12} - a^2{}_{21}) = 2D - s^2.$$

Hence $$(a \mid 12)\,(a \mid 23)\,(a \mid 31) = (3D - s^2)\,s \ldots\ldots\ldots (11).$$

Again,

$$(a \mid 12)\,(a \mid 23)\,(a \mid 34) = (a \mid 13)\,(a \mid 14)\,s - (13)\,(a \mid 34)\,D$$
$$= (a \mid 14)\,s^2 - (14)\,sD + (a \mid 14)\,D$$
$$= (a \mid 14)\,(D + s^2) - (14)\,sD;$$

whence also

$$(a \mid 12)\,(a \mid 23)\,(a \mid 34)\,(a \mid 41) = (2D - s^2)\,(D + s^2) + s^2D \ldots (12).$$

Suppose, in general, that

$$(a \mid 12)\,(a \mid 23) \ldots (a \mid lm) = A_m\,(a \mid 1m) - B_m\,(1m);$$

then

$$(a \mid 12)\,(a \mid 23) \ldots (a \mid mn) = A_m\,(a \mid 1m)\,(amn) + B_m\,(a \mid 1n)$$
$$= A_m\,\{(a \mid 1n)\,s - (1n)\,D\} + B_m\,(a \mid 1n)$$
$$= (sA_m + B_m)\,(a \mid 1n) - A_mD\,(1n).$$

Hence $$A_{m+1} = sA_m + B_m,\quad B_{m+1} = A_mD;$$

and consequently

$$A_{m+1} = sA_m + DA_{m-1} \ldots\ldots\ldots\ldots\ldots (13).$$

The form $\lambda_1\mu_1 + 0\lambda_1\mu_2 + s\lambda_2\mu_1 + D\lambda_2\mu_2$ is a form having s and D for invariants; and it may therefore be taken as the

canonical form for a bipartite quantic such as we have been here considering. Again, another useful form is the following,

$$\sqrt{(s^2 - 4D)}\,(\lambda_1\mu_2 + \lambda_2\mu_1) + s\,(\lambda\mu).$$

We may, however, combine not only two quadratic forms having like coefficients, say, two forms a; but we may also combine two having different coefficients, say, two forms a and b. Two such forms give rise to an invariant, namely, their product. In fact, we have

$$-(a\,|\,12)\,(b\,|\,12) = a_{11}b_{22} - a_{12}b_{21} - a_{21}b_{12} + a_{22}b_{11} = D_{ab},$$ suppose;

and from (5), $(a\,|\,12)\,(b\,|\,12) + (a\,|\,21)\,(b\,|\,12) = -s_a s_b$,

or
$$(a\,|\,21)\,(b\,|\,12) = D_{ab} - s_a s_b \quad \ldots\ldots\ldots\ldots\ldots\ldots (14).$$

The product of $(a\,|\,12) = a_{11}\lambda_1\mu_1 + \ldots$ and $(b\,|\,13) = b_{11}\lambda_1\nu_1 + \ldots$ gives a covariant; namely,

$$\begin{aligned}(a\,|\,12)\,(b\,|\,13) &= \begin{matrix} a_{11}\mu_1 + a_{12}\mu_2, & a_{21}\mu_1 + a_{22}\mu_2 \\ b_{11}\nu_1 + b_{12}\nu_2, & b_{21}\nu_1 + b_{22}\nu_2 \end{matrix} \\ &= \begin{vmatrix} a_{11}b_{21} - a_{21}b_{11}, & a_{11}b_{22} - a_{21}b_{12} \\ a_{12}b_{21} - a_{22}b_{11}, & a_{12}b_{22} - a_{22}b_{12} \end{vmatrix} (\mu_1, \mu_2)\,(\nu_1, \nu_2) \\ &= \mathfrak{H}_{ab}\,|\,23, \text{ suppose} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15).\end{aligned}$$

If, in the covariant, we make the forms a and b coincident, it becomes

$$(a\,|\,12)\,(a\,|\,13) = D_a(23), \quad (a\,|\,12)\,(a\,|\,23) = s_a\,(a\,|\,13) - D_a(13);$$

and, multiplying this by, or into, b, we get

$$(a\,|\,12)\,(a\,|\,13)\,(b\,|\,43) = (b\,|\,43)\,(a\,|\,12)\,(a\,|\,13) = -D_a\,(b\,|\,42),$$

$$(a\,|\,12)\,(a\,|\,13)\,(b\,|\,34) = (b\,|\,43)\,(a\,|\,12)\,(a\,|\,13) = -D_a\,(b\,|\,24),$$

and
$$(a\,|\,12)\,(a\,|\,13)\,(b\,|\,23) = D_a s_b,$$

$$(a\,|\,12)\,(a\,|\,13)\,(b\,|\,32) = D_a s_b \quad \ldots\ldots\ldots\ldots\ldots (16).$$

The remaining form to be investigated is

$$(a\,|\,12)\,(b\,|\,13)\,(a\,|\,43).$$

The value of this is

$$(a_{11}b_{21}-a_{21}b_{11})\mu_1+(a_{12}b_{21}-a_{22}b_{11})\mu_2,\quad (a_{11}b_{22}-a_{21}b_{12})\mu_1+(a_{12}b_{22}-a_{22}b_{12})\mu_2$$
$$a_{11}\rho_1+\qquad a_{21}\rho_2,\qquad a_{12}\rho_1+\qquad a_{22}\rho_2$$
$$\begin{aligned}
&= (-a_{21}a_{12}b_{11}+a_{11}a_{21}b_{12}+a_{11}a_{12}b_{21}-a_{11}a_{11}b_{22})\,\mu_1\rho_1\\
&\quad+(-a_{21}a_{22}b_{11}+a_{21}a_{21}b_{12}+a_{11}a_{22}b_{21}-a_{11}a_{21}b_{22})\,\mu_1\rho_2\\
&\quad+(\ a_{12}a_{12}b_{21}-a_{22}a_{12}b_{21}-a_{11}a_{12}b_{22}+a_{11}a_{22}b_{12})\,\mu_2\rho_1\\
&\quad+(\ a_{22}a_{12}b_{21}-a_{22}a_{22}b_{11}-a_{21}a_{12}b_{22}+a_{21}a_{22}b_{12})\,\mu_2\rho_2\\
&= (-a_{11}D_{ab}+b_{11}D_{aa})\,\mu_1\rho_1\\
&\quad+(-a_{21}D_{ab}+b_{21}D_{aa})\,\mu_1\rho_2\\
&\quad+(-a_{12}D_{ab}+b_{12}D_{aa})\,\mu_2\rho_1\\
&\quad+(-a_{22}D_{ab}+b_{22}D_{aa})\,\mu_2\rho_2\\
&= D_{ab}(a\,|\,42)-D_{aa}(b\,|\,42)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (17).
\end{aligned}$$

Multiplying this into $(b\,|\,45)$, we obtain

$$(a\,|\,12)\,(b\,|\,13)\,(a\,|\,43)\,(b\,|\,45)$$
$$= D_{ab}(a\,|\,42)\,(b\,|\,45)-D_{aa}(b\,|\,42)\,(b\,|\,45);$$

that is, referring to equation (15), and dropping the suffix ab,

$$(\mathfrak{S}\,|\,23)\,(\mathfrak{S}\,|\,35)=D_{ab}(\mathfrak{S}\,|\,25)-D_{aa}.D_{bb}(25).$$

But $$(\mathfrak{S}\,|\,23)\,(\mathfrak{S}\,|\,35)=s\,(\mathfrak{S}_2\,|\,25)-D_2(25).$$

Hence $$s_2=D_{ab},\quad D_2=D_{aa}.D_{bb}\ldots\ldots\ldots\ldots\ldots\ldots(18),$$

as may be easily verified.

If, however, we multiply the equation

$$(a\,|\,12)+(a\,|\,21)=-s\,(12)\text{ by }(a\,|\,12),$$

we obtain

$$(a\,|\,12)\,(a\,|\,21)=2D_a-s_a^2;$$

hence $$(\mathfrak{S}\,|\,12)\,(\mathfrak{S}\,|\,21)=2D_2-s_2^2=2D_aD_b-D_{ab}^2\ \ldots\ldots\ (19),$$

and consequently

$$\left.\begin{aligned}\{(\mathfrak{S}\,|\,12)-(\mathfrak{S}\,|\,21)\}^2&=-4D_aD_b-2\,(2D_aD_b-D_{ab}^2)\\&=-2\,(4D_aD_b-D_{ab}^2)\end{aligned}\right\}\ldots.(20).$$

The formula $s_2=D_{ab}$ is important as showing that $\mathfrak{S}$ is not made symmetrical by making a and b symmetrical. Hence, in passing from these invariants and covariants to those of sym-

metrical forms, we are obliged to use the symmetrised form of ϑ, namely $(\vartheta \mid 12) - (\vartheta \mid 21)$. Thus, to adapt equation (17) to symmetrical forms, we have

$$(\vartheta \mid 23)\,(a \mid 43) = D_{ab}\,(a \mid 42) - D_{aa}\,(b \mid 42),$$

$$\begin{aligned}(\vartheta \mid 23)\,(a \mid 43) &= (a \mid 13)\,(b \mid 12)\,(a \mid 43)\\ &= D_a\,(14)\,(b \mid 12) = -D_a\,(b \mid 42).\end{aligned}$$

In general, we write $\bar{a}$ for the mean value of a, i.e., if

$$2\,(\bar{a} \mid 12) = (a \mid 12) - (a \mid 21),$$

we shall have $$(\bar{a} \mid 12) = (a \mid 12) + \tfrac{1}{2}s\,(12).$$

Hence, since $(12)^2 = -2$,

$$(\bar{a} \mid 12)^2 = (a \mid 12)^2 + s^2 - \tfrac{1}{2}s^2 \ldots\ldots\ldots\ldots\ldots (21),$$

and $$\overline{D} = D - \tfrac{1}{4}s^2,$$

where $\overline{D}$ is the discriminant of the symmetrical function.

Similarly

$$\begin{aligned}\vartheta_{\bar{a}\bar{b}} = (\bar{a} \mid 12)\,(\bar{b} \mid 13) &= \{(a \mid 12) + \tfrac{1}{2}s_a\,(12)\}\,\{(b \mid 13) + \tfrac{1}{2}s_b\,(13)\}\\ &= \vartheta_{ab} - \tfrac{1}{2}\{s_a\,(b \mid 23) + s_b\,(a \mid 23)\} + \tfrac{1}{4}s_a s_b\,(23),\end{aligned}$$

and therefore

$$\left.\begin{aligned}2\bar{\vartheta}_{\bar{a}\bar{b}} &= (\vartheta_{ab} \mid 23) - (\vartheta_{ab} \mid 32) - s_a s_b\,(23)\\ &= 2\,(\vartheta_{ab} \mid 23) - s_b\,(23) - s_a s_b\,(23)\end{aligned}\right\} \ldots\ldots\ldots (22).$$

If we multiply this into $(a \mid 43)$, we have

$$\begin{aligned}2\,(\vartheta_{\bar{a}\bar{b}} \mid 23)\,(a \mid 43) &= 2\,(\vartheta \mid 23)\,(a \mid 43) - D_{ab}\,(a \mid 42) - s_a s_b\,(a \mid 42)\\ &= D_{ab}\,(a \mid 42) - 2D_a D_b\,(b \mid 42) - s_a s_b\,(a \mid 42).\end{aligned}$$

Putting $s_a = 0$, $s_b = 0$, we get the formula for symmetric functions.

XXX.

APPLICATIONS OF GRASSMANN'S EXTENSIVE ALGEBRA*.

I PROPOSE to communicate in a brief form some applications of Grassmann's theory which it seems unlikely that I shall find time to set forth at proper length, though I have waited long for it. Until recently I was unacquainted with the *Ausdehnungslehre,* and knew only so much of it as is contained in the author's geometrical papers in *Crelle's Journal* and in *Hankel's Lectures on Complex Numbers.* I may, perhaps, therefore be permitted to express my profound admiration of that extraordinary work, and my conviction that its principles will exercise a vast influence upon the future of mathematical science.

The present communication endeavours to determine the place of Quaternions and of what I have elsewhere† called Biquaternions in the more extended system, thereby *explaining* the laws of those algebras in terms of simpler laws. It contains, next, a generalization of them, applicable to any number of dimensions; and a demonstration that the algebra thus obtained is always a compound of quaternion algebras which do not interfere with one another.

On the Relation of Grassmann's Method to Quaternions and Biquaternions; and on the Generalization of these Systems.

Following a suggestion of Professor Sylvester, I call that kind of multiplication in which the sign of the product is reversed by an interchange of two adjacent factors, *polar* multi-

* [*American Journal of Mathematics Pure and Applied,* Vol. I. pp. 350—358.]

† *Proceedings of the London Mathematical Society* [*XX. supra*].

plication*; because the product ab has opposite properties at its two ends, so that $ab = -ba$. The ordinary or commutative multiplication I shall call *Scalar*, being that which holds good of scalar numbers. These words answer to Grassmann's *outer* and *inner* multiplication; which names, however, do not describe the multiplication itself, but rather those geometrical circumstances to which it applies.

Consider now a system of n units $\iota_1, \iota_2, \ldots \iota_n$, such that the multiplication of any two of them is polar; that is, $\iota_r \iota_s = -\iota_s \iota_r$. For geometrical applications we may take these to represent points lying in a flat space of $n-1$ dimensions. A binary product $\iota_r \iota_s$ is then a unit length measured on the line joining the points ι_r, ι_s; a ternary product $\iota_r \iota_s \iota_t$ is a unit area measured on the plane through the three points, and so on. A linear combination of these units, $\Sigma a_r \iota_r = \alpha$ suppose, represents a point in the given flat space of $n-1$ dimensions, according to the principles of the barycentric calculus, as extended in the Ausdehnungslehre of 1844.

In space of three dimensions we may take the four points $\iota_0, \iota_1, \iota_2, \iota_3$ so that $\iota_1, \iota_2, \iota_3$ are at an infinite distance from ι_0 in three directions at right angles to one another.

Now there are two sides to the notion of a product. When we say $2 \times 3 = 6$, we may regard the product 6 as a number derived from the numbers 2 and 3 by a process in which they play similar parts; or we may regard it as derived from the number 3 by the operation of doubling. In the former view 2 and 3 are both numbers; in the latter view 3 is a number, but 2 is an operation, and the two factors play very distinct parts. *The Ausdehnungslehre is founded on the first view; the theory of quaternions on the second.* When a line is regarded as the product of two points, or a parallelogram as the product of its sides, the two factors are things of the same kind and play similar parts. But in such a quaternion equation as $q\rho = \sigma$, where ρ and σ are vectors, the quaternion q is an operation of turning and stretching which converts ρ into σ; it is a thing totally different in kind from the vector ρ. The only way in

* [*American Journal of Mathematics*, Vol. I. p. 127 and p. 257, *supra*.]

which the factors q and ρ can be taken to be of the same kind, is to regard ρ as itself a special case of a quaternion, viz. a rectangular versor. But in that case the expression does not receive its full meaning until we suppose a *subject* on which the operations ρ and q can be performed in succession.

The quaternion symbols i, j, k represent, then, *rectangular versors;* that is to say, they are operations which will turn a figure through a right angle in the three co-ordinate planes respectively. It follows that if either of them is applied twice over to the same figure, it will turn it through two right angles, or *reverse* it; we must therefore have $i^2 = j^2 = k^2 = -1$.

To compare these with the symbols for the four points $\iota_0, \iota_1, \iota_2, \iota_3$, let us suppose that i turns the line $\iota_0 \iota_2$ into $\iota_0 \iota_3$; that j turns $\iota_0 \iota_3$ into $\iota_0 \iota_1$; and that k turns $\iota_0 \iota_1$ into $\iota_0 \iota_2$. The turning of $\iota_0 \iota_2$ into $\iota_0 \iota_3$ is equivalent to a translation along the line at infinity $\iota_2 \iota_3$. We may, therefore, write $i = \iota_2 \iota_3$, and so $j = \iota_3 \iota_1$, $k = \iota_1 \iota_2$. Now i turns $\iota_0 \iota_2$ into $\iota_0 \iota_3$; that is

$$i \,.\, \iota_0 \iota_2 = \iota_0 \iota_3,$$

or

$$\iota_0 \iota_3 = \iota_2 \iota_3 \,.\, \iota_0 \iota_2 = -\iota_2^2 \,.\, \iota_0 \iota_3.$$

We are therefore obliged to write $\iota_2^2 = -1$, and in a similar way we may find $\iota_1^2 = \iota_3^2 = -1$.

This at once enables us to find the rules of multiplication of the i, j, k. Namely, we have

$$jk = \iota_3 \iota_1 \,.\, \iota_1 \iota_2 = \iota_2 \iota_3 = i,$$

$$ki = \iota_1 \iota_2 \,.\, \iota_2 \iota_3 = \iota_3 \iota_1 = j,$$

$$ij = \iota_2 \iota_3 \,.\, \iota_3 \iota_1 = \iota_1 \iota_2 = k,$$

and finally

$$ijk = \iota_2 \iota_3 \,.\, \iota_3 \iota_1 \,.\, \iota_1 \iota_2 = -1.$$

In order, therefore, to bring the quaternion algebra within that of the Ausdehnungslehre, we have to make the square of each of our units equal to -1, as pointed out by Grassmann (*Math. Annalen*). But I venture to differ from his authority in thinking that the quaternion symbols do not in the first place answer to the "Elementargrösse" of the Ausdehnungslehre, but to

binary products of them; from which supposition, as we have seen, the laws of their multiplication follow at once.

It is quite true that in process of time the conception of a product as derived from factors of the same kind, and so of the product of two vectors, as a thing which might be thought of without regarding them as rectangular versors, grew upon Hamilton's mind, and led to the gradual replacement of the units i, j, k by the more general selective symbols S and V. To explain the laws of multiplication of i, j, k on this view, we must have recourse to the theory of "Ergänzung," or which comes to the same thing, *represent* an area ij by a vector k perpendicular to it. But the explanation in this case is by no means so easy; and it is instructive to observe that the distinction between a quantity and its "Ergänzung," i.e. between an area and its representative vector, which, for some purposes, it is so convenient to ignore, has to be reintroduced in physics. Thus Maxwell specially distinguishes the two kinds of vectors, which he calls *force* and *flow*, and which in fact are respectively linear functions of the units and of their binary products.

We have regarded the symbols i, j, k as rectangular versors operating on the quantities $\iota_0\iota_1$, $\iota_0\iota_2$, $\iota_0\iota_3$. These quantities are unit lengths measured anywhere on the axes in the positive directions. They have magnitude, direction, and position, and are thus what I have called *rotors* (short for *rotators*) to distinguish them from *vectors*, which have magnitude and direction but no position. A vector is of the nature of the translation-velocity of a rigid body, or of a couple; it may be represented by a straight line of given length and direction drawn *anywhere*. A rotor is of the nature of the rotation-velocity of a rigid body, or of a force; it belongs to a definite axis. A vector may be represented as the difference of two points of equal weight (the vector ab may be written $b-a$); this is shewn by the principles of the barycentric calculus to represent a point of no weight at infinity. Accordingly the symbols $\iota_1, \iota_2, \iota_3$ may be taken to mean unit vectors along the axes. In fact, if we write $\iota_0 + \iota_r = \alpha_r$, the points α will be situate on the axes at unit distance from the origin, and thus $\iota_r = \alpha_r - \iota_0$ will represent the unit vector from the origin to α_r.

The versors i, j, k will operate on these vectors in the same way as on the rotors $\iota_0\iota_1$, $\iota_0\iota_2$, $\iota_0\iota_3$. We find that

$$i\iota_2 = \iota_2\iota_3 \cdot \iota_2 = \iota_3, j\iota_3 = \iota_1, k\iota_1 = \iota_2.$$

These rules of multiplication coincide with those for i, j, k if we write the latter in place of ι_1, ι_2, ι_3. Thus we may use the same symbols to represent unit vectors along the axes and rectangular versors about them. But it is not in any sense true that the vectors ι_1, ι_2, ι_3 are identical with the areas $\iota_2\iota_3$, $\iota_3\iota_1$, $\iota_1\iota_2$; it is only sometimes convenient to forget the difference between ι_1 and $\iota_2\iota_3$.

In the elliptic or hyperbolic geometry* of three dimensions, the four points ι_0, ι_1, ι_2, ι_3 must be taken as the vertices of a tetrahedron self-conjugate in regard to the absolute, so that the distance between every two of them is a *quadrant*. The product of four points $\alpha\beta\gamma\delta$ will then consist of three kinds of terms; (1) terms of the fourth order, being $\iota_0\iota_1\iota_2\iota_3$ multiplied by the determinant of the co-ordinates of the four points, which is proportional to $\sin(\alpha, \beta) \sin(\gamma, \delta) \cos(\alpha\beta, \gamma\delta)$; (2) terms of the second order, resulting from products of the form $\iota_0^2\iota_1\iota_2 = -\iota_1\iota_2$; (3) terms of order zero, resulting from products of the form $\iota_0^4\iota_0^2\iota_1^2$. Altogether we may arrange $\alpha\beta\gamma\delta$ in eight terms as follows:

$$\alpha\beta\gamma\delta = a + \Sigma b_{rs}\iota_r\iota_s + c\iota_0\iota_1\iota_2\iota_3. \quad (r, s \text{ different.})$$

And it is now easy to see that the product of any *even* number of linear factors will be of the same form. This form is what I have called a *biquaternion*, and may be easily exhibited as such. Namely, let us write ω for $\iota_0\iota_1\iota_2\iota_3$; then we have

$$i = \iota_2\iota_3, \; j = \iota_3\iota_1, \; k = \iota_1\iota_2,$$

$$\omega i = i\omega = \iota_1\iota_0, \; \omega j = j\omega = \iota_2\iota_0, \; \omega k = k\omega = \iota_3\iota_0,$$

$$\omega^2 = 1.$$

Therefore, the product of any even number of factors greater than two is a linear function of $1, i, j, k, \omega, \omega i, \omega j, \omega k$; that is to say, it is of the form $q + \omega r$, where q, r are quaternions.

* Dr Klein's names for the Geometry of a space of uniform positive or negative curvature. [Cf. p. 191 *supra*.]

While the multiplication of ω with i, j, k is scalar, its multiplication with $\iota_0, \iota_1, \iota_2, \iota_3$ is polar. The effect of multiplying by ω is to change any system into its polar system in regard to the absolute.

The chief classification of geometric algebras is into those of *odd* and *even* dimensions. The geometry of an elliptic space of n dimensions is the same as the geometry of the points at an infinite distance in a flat or parabolic space of $n + 1$ dimensions; the theory of *points* and *rotors* in the former is the same as that of vectors and their products in the latter. Each requires a geometric algebra of $n + 1$ units. Thus the algebra of four units, leading as above to biquaternions, is either that of points and rotors in an elliptic space of three dimensions, or of vectors and their products in a flat space of four dimensions. All geometric algebras having an even number of units are closely analogous to it; of these I would point out particularly that of two units, belonging to the elliptic geometry of one dimension or to the theory of vectors in a plane. Let the units be ι_2, ι_3; then a product of any even number of linear functions must be of the form $a + b\iota_2\iota_3$. Let $i = \iota_2\iota_3$, then $i^2 = -1$; and such an even product is the ordinary complex number $a + bi$. In the method of Gauss every vector in the plane is represented by means of its ratio to the unit vector ι_2, that is to say, ι_2 and ι_3 are replaced by 1 and i. This gives an artificial but highly useful value for the product of two vectors. We might apply a similar interpretation to the algebra of four units, denoting the points $\iota_0, \iota_1, \iota_2, \iota_3$ by the symbols ω, i, j, k, and consequently their polar planes $\omega\iota_0, \omega\iota_1, \omega\iota_2, \omega\iota_3$ by the symbols 1, ωi, ωj, ωk, but I am not aware that any useful results would follow from this imitation of Gauss's plane of numbers.

Rules of Multiplication in an Algebra of n *units.*

In general, if we consider an algebra of n units, $\iota_1, \iota_2, \ldots \iota_n$, such that $\iota_r^2 = -1$, $\iota_r\iota_s = -\iota_s\iota_r$, a product of m linear factors will contain terms which are all of even order if m is even, and all of odd order if m is odd; for the substitution of -1 for any square factor of a term reduces the order of the term by 2.

A product of m units, all different, multiplied by any scalar is called a *term* of the order m. The sum of several terms of order m each multiplied by a scalar, is a *form* of order m. The sum of several forms of different orders is a *quantity* and an even quantity when the forms are all of even order, an odd quantity when they are all of odd order. Thus the multiplication of linear functions of the units leads only to even quantities and odd quantities.

The square of a term of the mth *order is* $+1$ *or* -1 *according as the integer part of* $\frac{1}{2}(m+1)$ *is even or odd.* For the product $\iota_1\iota_2 \dots \iota_m\iota_1\iota_2 \dots \iota_m$ is transformed into $\iota_1^2\iota_2^2 \dots \iota_m^2$ by $\frac{1}{2}m(m-1)$ changes of consecutive factors, and therefore equals ± 1 according as $\frac{1}{2}m(m+1)$ is even or odd, which is equivalent to the rule stated.

The multiplication of a term P *of order* m *by a term* Q *of order* n, *having* k *factors common, is scalar or polar according as* $\mathrm{mn}-\mathrm{k}^2$ *is even or odd.* Let $P=CP'$ and $Q=CQ'$, where C, P', Q' have no common factor; then the steps from $CP'CQ'$ to $CP'Q'C$, $CQ'P'C$, $CQ'CP'$ require respectively $k(n-k)$, $(m-k)(n-k)$, $k(m-k)$ changes of consecutive factors; and the sum of these quantities is even or odd as $mn-k^2$ is.

The following cases are worth noticing:

(1) When two terms have no factor common, their multiplication is scalar except when they are both of odd order. (Case $k=0$.)

(2) The multiplication of two even terms is scalar or polar according as the number of common factors is even or odd.

(3) If one of two terms is a factor in the other, the multiplication is scalar except when the first is odd and the second even.

Theory of Algebras with an odd number of units.

When the number of units is $n=2m+1$, there are n terms of the order $n-1$, and all terms of even order can be expressed by means of these. For the product of any two of these terms is of the second order, since they must have $n-2$ factors common.

We obtain in this way all the terms of the second order; and from them we can build up the terms of the fourth, sixth orders, &c. Let the product of all the units $\iota_1\iota_2 \dots \iota_n$ be called ω, then these terms of the order $n-1$ shall be defined by the equations $k_r = \omega\iota_r$. It will follow that $k_1k_2 \dots k_n = \mp 1$ according as m is even or odd, or, what is the same thing, according as the squares of the k are $+1$ or -1. By means of this formula, terms of order higher than m in the k, may be replaced by terms of order not higher than m. The multiplication of the k is always polar.

The terms of even order, regarded as compound units, constitute an algebra which is *linear* in the sense of Professor Peirce, viz. it is such that the product of any two of these terms is again a term of the system. The number of them is $2^{n-1} = 2^{2m}$; for the whole number of terms, odd and even, is

$$1 + n + \tfrac{1}{2}n(n-1) + \dots + n + 1 = (1+1)^n = 2^n,$$

and the number of even terms is clearly equal to the number of odd terms.

I shall call the algebras whose units are the even terms formed with n elementary units $\iota_1\iota_2 \dots \iota_n$, the n-*way geometric algebra*. Thus quaternions are the *three-way* algebra. We may regard the units of quaternions as expressed in either of two ways. First, in terms of the elementary units $\iota_1\iota_2\iota_3$; they are then $(1, \iota_2\iota_3, \iota_3\iota_1, \iota_1\iota_2)$. Secondly, we may write k_1, k_2 for the terms $\iota_2\iota_3$, $\iota_3\iota_1$, and the system may then be written $(1, k_1, k_2, k_1k_2)$. In this second form it is identical with the entire algebra of two elementary units, including both odd and even terms.

The five-way algebra depends upon the five terms k_1, k_2, k_3, k_4, k_5 and their products; the number of terms is sixteen. Now *we may obtain the whole of these sixteen terms by multiplying the quaternion set*

$$(1, k_1, k_2, k_1k_2)$$

by this other quaternion set

$$(1, k_4k_5, k_5k_3, k_3k_4).$$

For each of the sixteen products so obtained is a term of the even five-way algebra, and the products are all distinct. More-

over, *the two quaternion sets are commutative with one another.* For since the k multiply in the polar manner, we may regard them as elementary units for this purpose; now the terms in the second set are all even, and no term in one set has a factor common with any term in the other set.

In the language of Professor Peirce, then, the five-way algebra is a compound of two quaternion algebras, which do not in any way interfere, because the units of one are commutative in regard to those of the other. A quantity in the five-way algebra is in fact a quaternion $\omega + ix + jy + kz$, whose coefficients ωxyz are themselves quaternions of another set of units $(1, i_1, j_1, k_1)$, the i_1, j_1, k_1 being commutative with i, j, k.

I shall now extend this proposition, and shew that *the* $(2m+1)$ *way algebra is a compound of* m *quaternion algebras, the units of which are commutative with one another.* To this end let us write $p_0 = k_1 k_2$, and then

$$\begin{aligned} p_1 &= k_1 k_2 k_6 k_7 = p_0 k_6 k_7, & q_1 &= k_3 k_4 k_5, \\ p_2 &= p_1 k_{10} k_{11} \quad , & q_2 &= q_1 k_8 k_9, \\ &\cdots\cdots\cdots\cdots\cdots\cdots & &\cdots\cdots\cdots \\ p_r &= p_{r-1} k_{4r+2} k_{4r+3} \quad , & q_r &= q_{r-1} k_{4r} k_{4r+1}. \end{aligned}$$

Consider now the quaternion sets

$$\begin{aligned} &1, k_1, k_2, k_1 k_2 \\ &1, k_4 k_5, k_5 k_3, k_3 k_4 \\ &1, p_0 k_6, p_0 k_7, k_6 k_7 \\ &1, q_1 k_8, q_1 k_9, k_8 k_9 \\ &1, p_1 k_{10}, p_1 k_{11}, k_{10} k_{11} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &1, q_{r-1} k_{4r}, q_{r-1} k_{4r+1}, k_{4r} k_{4r+1} \\ &1, p_{r-1} k_{4r+2}, p_{r-1} k_{4r+3}, k_{4r+2} k_{4r+3}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

viz.: a p-set and a q-set alternately. I say that if we consider the first m sets of this series, we shall find them to involve $2m+1$ of the k; that the products of m terms, one from each series, constitute 2^{2m} distinct terms, which are therefore identical

with the terms of the $(2m+1)$ way algebra; and that the terms in any two sets are commutative with each other. The first two remarks are obvious on inspection; the last also is clear for the case of a p-set and a q-set, because the q-set is of even order in the k, and no factors are common to the two sets. It remains only to examine the case of two p-sets and of two q-sets. Compare the two p-sets

$$1,\ p_{r-1}k_{4r+2},\ p_{r-1}k_{4r+3},\ k_{4r+2}k_{4r+3},$$
$$1,\ p_{s-1}k_{4s+2},\ p_{s-1}k_{4s+3},\ k_{4s+2}k_{4s+3},$$

where $s > r$. All the terms of the first set are contained as factors in each of the terms $p_{s-1}k_{4s+2}$, $p_{s-1}k_{4s+3}$, which are of odd order in the k; consequently the multiplication is scalar. The term $k_{4s+2}k_{4s+3}$ has no factor common with the first set, and being of even order is commutative in regard to it. Hence the two sets are commutative with one another. Next take the two q-sets

$$1,\ q_{r-1}k_{4r},\ q_{r-1}k_{4r+1},\ k_{4r}k_{4r+1},$$
$$1,\ q_{s-1}k_{4s},\ q_{s-1}k_{4s+1},\ k_{4s}k_{4s+1}.$$

Here again all the terms of the first sets are factors of $q_{s-1}k_{4s}$ and of $q_{s-1}k_{4s+1}$, and they have no factors in common with $k_{4s}k_{4s+1}$; since then all the terms are of even order in the k, the multiplication is scalar. The proposition is therefore proved.

We may set out a formal proof that the 2^{2m} products of m terms, one from each of the first m sets, are all *distinct*, as follows: suppose this true for the first $m-1$ sets; that is to say, that no two of the products formed from them are identical or such that their product is $\pm k_1k_2 \dots k_{2m-1}$. Let then a, b be two of these products; and let c, d be two terms of the next set. Then we have to prove that ac can neither be equal to $\pm bd$, nor such that the product $acbd$ is $\pm k_1k_2 \dots k_{2m-1}k_{2m}k_{2m+1}$. Now if $ac = \pm bd$, multiply both sides by bc; then $ab = \pm cd$. The product cd is one of the terms of the new set; it is either unity, or contains one or both of the new units k_{2m}, k_{2m+1}, so that it cannot be equal to ab. The product $abcd$ cannot be $\pm k_1 \dots k_{2m+1}$ unless cd is $k_{2m}k_{2m+1}$ and ab is $k_1k_2 \dots k_{2m-1}$, which is contrary to the supposition. Hence if the products of the first

$m-1$ sets are all distinct for the purposes of the $(2m-1)$ way algebra, the products of the first m sets will be all distinct for the purposes of the $(2m+1)$ way algebra. But it is easy to see that the products of the first two sets are distinct.

Algebras with an even number of units.

Every algebra with $2m$ units is related to the adjacent algebra with $2m-1$ units in precisely the same way as biquaternions are related to quaternions; namely, it is simply that adjacent algebra multiplied by the double algebra $(1, \omega)$ where ω is the product of all the $2m$ units. For clearly all the even terms of the $(2m-1)$ way algebra are also even terms of the $2m$-way algebra, and so also are their products by ω; but these are all distinct from one another, and consequently are *all* the even terms of the $2m$-way algebra.

The multiplication of ω with the k of the $(2m-1)$ way algebra is scalar, because the k are factors in the ω, and they are both even terms.

Hence the $2m$-way algebra is a product of the $(2m-1)$ way algebra with the double algebra $(1, \omega)$, the two sets of units being commutative with one another.

*XXXI.

BINARY FORMS OF ALTERNATE VARIABLES*.

INTRODUCTION.

1. ALTERNATE numbers are such that $\alpha\beta = -\beta\alpha$, $\alpha^2 = 0$, $\beta^2 = 0$. It is easily shown that linear functions of them possess the same properties; i.e., if $\bar{\alpha} = a_1\alpha_1 + a_2\alpha_2 + \ldots$, $\bar{\beta} = b_1\beta_1 + b_2\beta_2 + \ldots$, where the a, b are scalars, and the α, β alternate numbers, then we shall have $\bar{\alpha}\bar{\beta} = -\bar{\beta}\bar{\alpha}$, and $\bar{\alpha}^2 = 0 = \bar{\beta}^2$. If M, N are homogeneous functions of alternate numbers of degrees m, n respectively, the number of interchanges of consecutive letters necessary to pass from MN to NM is mn; thus we have

$$MN = (-)^{mn} NM.$$

Or the product of two functions changes sign when the order of the functions is changed *only when their degrees are both odd;* that is to say, forms of odd degree among themselves behave like alternate numbers, forms of even degree in all cases like scalars. It follows that the square of any form of odd degree is zero.

Determinants of alternate numbers.

2. In expanding a determinant of alternate numbers, the order of the *rows* must be followed in multiplication; that is to say, in every term of the expanded determinant the

* [Communicated to the London Mathematical Society (June 12, 1879) by Dr Spottiswoode, P.R.S., and subsequently printed in the *Proceedings*, Vol. x. pp. 214—221. See note at end of this paper.]

constituent from the first row must be written first, that from the second row second, and so on. Thus, in expanding

$$\begin{vmatrix} \lambda_1, & \lambda_2, & \lambda_3 \\ \mu_1, & \mu_2, & \mu_3 \\ \nu_1, & \nu_2, & \nu_3 \end{vmatrix} = (\lambda\mu\nu),$$

the terms are of the form $\pm \lambda_r \mu_s \nu_t$, where rst is a permutation of 123 and the signs follow the ordinary rule. An interchange of two *columns* will then alter the sign of the determinant but an interchange of two rows will leave it unaltered. For the change of sign caused by the interchange is in the latter case counteracted by the change in the order of multiplication. Thus the determinant $(\lambda\mu\nu)$ written above is a symmetrical function of the $\lambda\mu\nu$.

3. Alternate numbers may be considered as given in *sets* of n at a time (like the coordinates of a point in n-fold space), and in that case it is convenient to regard the product of all the numbers of any set as equal to unity. Hence the products of all but one of the numbers make a new set, the *reciprocal* numbers. The nth power of a determinant of *even* order n is $-\lfloor n$; the $(n-1)^{\text{th}}$ power is the determinant of the reciprocal numbers. Considering especially determinants of the second order, we have an important theorem of their multiplication, viz., $(\lambda\mu)(\mu\nu) = -(\lambda\nu)$. For

$$\begin{aligned} (\lambda_1\mu_2 - \lambda_2\mu_1)(\mu_1\nu_2 - \mu_2\nu_1) &= \lambda_1\mu_2\mu_1\nu_2 + \lambda_2\mu_1\mu_2\nu_1 \\ &= (\lambda_2\nu_1 - \lambda_1\nu_2)\mu_1\mu_2 \\ &= -(\lambda_1\nu_2 - \lambda_2\nu_1). \end{aligned}$$

Hence $(\lambda\mu)(\mu\nu)(\nu\rho)\ldots(\sigma\tau)$ $\{n \text{ factors}\} = (-)^{n-1}(\lambda\tau)$.

An analogous theorem holds for determinants of the nth order; viz., if we denote a determinant with r rows of λ and s rows of μ by $(\lambda^r\mu^s)$, where $r+s=n$; then

$$(\lambda^r\mu^s)(\mu^r\nu^s)\ldots(\sigma^r\tau^s)\{k \text{ factors}\} = (-)^{s(n+r)k}(\lambda^r\tau^s) = (-)^{sk}(\lambda^r\tau^s),$$

since $n + r = 2r + s \equiv$ (mod. 2), and so $s(n+r) \equiv s^2 \equiv s$ (mod. 2).

Multipartite Forms.

4. A homogeneous function linear in each of n sets of k alternate numbers is called a k-ary multipartite form of the nth order, or shortly, a k-ary form of the nth order. We may consider also forms of any order lower than k in any of the sets; but for the present we restrict ourselves to the case in which the forms are linear. Consider now k forms, each linear in regard to the k alternate numbers $\lambda_1, \lambda_2, \ldots \lambda_k$; viz.,

$$F_a = a_1\lambda_1 + a_2\lambda_2 + \ldots + a_k\lambda_k$$
$$\vdots$$
$$F_k = k_1\lambda_1 + k_2\lambda_2 + \ldots + k_k\lambda_k,$$

where the coefficients $a, b, \ldots k$ are themselves k-ary multipartite forms of alternate numbers.

The product $F_aF_b \ldots F_k = \Pi F$ is an invariant; that is to say, if for the λ we substitute k linear functions of them, say the μ, then the functions F will be transformed into functions of the μ; and if we form the same function of the new coefficients that ΠF is of the old coefficients, one will be equal to the other multiplied by the determinant of transformation.

For the product is $I.\lambda_1\lambda_2 \ldots \lambda_k$, whether we regard the λ as linear functions of the μ or not; but in the latter case

$$\lambda_1\lambda_2 \ldots \lambda_k = D.\mu_1\mu_2 \ldots \mu_k,$$

where D is the determinant of transformation.

But it is also to a constant factor *près* the only function possessing this property. For let I be such a function, and calculate it for the linear forms $\lambda_1, \lambda_2, \ldots \lambda_k$. As all the coefficients are here either unity or zero, I must be represented by a constant, say I_0. Now, expressing the λ in terms of the μ, we have, by hypothesis, $I_\mu = DI_0 = \lambda_1 \ldots \lambda_k . I_0$.

It is to be remarked that the coefficients of transformation may themselves be forms involving alternate numbers to any *even* order. Otherwise the μ would not be alternate numbers, which is implied.

Moreover, we may regard the λ as no longer linear forms, but forms of any odd order; the new invariant will then be equal to the old one multiplied by a *commutant* of transformation. This leads to a useful theorem in the comparison of invariants; e.g.,

$$c\,|\,456\theta\,.\,a\,|\,\theta 2\,.\,b\,|\,\theta' 3\,.\,c\,|\,456\theta' = (c\,|\,456\theta\,|)^2\,.\,a\,|\,\theta' 2\,.\,b\,|\,\theta' 3.$$

The proposition may be further extended by considering forms which involve the λ to an order higher than the first, but less than k; i.e., linear functions of their products r together. Let F_a, F_b, ...F_k be forms such that the sum of their orders in the λ is equal to k; then their product is an invariant (and the only one) in regard to linear transformations of the λ.

If the sum of the orders is less than k, $= h$ suppose, the product is a covariant; viz., it is a linear function of the products of the λ, h together, which, whether derived from the original forms before or after the λ are replaced by linear functions of the μ, has the same value. In this case the r-products of the λ are replaced by linear functions of the r-products of the μ, the coefficients being determinants of the rth order formed with the coefficients of the μ.

Now suppose any number of forms $F_a \ldots F_n$ to involve any number of sets of alternate numbers $\lambda\mu \ldots \tau$, yet so that the sum of the orders in any one set is not greater than the number of alternates in that set; then the product of the forms is an invariant or covariant in regard to each of the sets taken separately, in the sense explained above; and it is the only function which possesses this property.

Expression of Unsymmetrical forms in terms of Symmetrical forms and Determinants of the Variables.

5. The binary form in two sets of variables,

$$a_{11}\lambda_1\mu_1 + a_{12}\lambda_1\mu_2 + a_{21}\lambda_2\mu_1 + a_{22}\lambda_2\mu_2 = a12,$$

will be called *symmetrical* when $a_{12} = a_{21}$. In that case the

interchange of the variables λ, μ only alters the sign of the form; we have $a12 = -a21$. We have, in general,

$$a12 + a21 = (a_{12} - a_{21})(\lambda_1\mu_2 - \lambda_2\mu_1) = s(\lambda\mu).$$

The factor $a_{12} - a_{21}$ is an invariant when both sets of variables are transformed by the same substitutions; in fact, we have

$$(\lambda_1\mu_2 - \lambda_2\mu_1)\, a12 = a_{21} - a_{12} = -s,$$

which exhibits it as a product of the form by the universal covariant $(\lambda\mu)$ or (12). We may make a symmetrical form from $a12$ by adding $-a21$ to it; half this sum shall be called the mean value of $a12$ and denoted by $\overline{a12}$. Thus, we have

$$\overline{a12} = \tfrac{1}{2}(a12 - a21) = a_{11}\lambda_1\mu_1 + \tfrac{1}{2}(a_{12} + a_{21})(\lambda_1\mu_2 + \lambda_2\mu_1) + a_{22}\lambda_2\mu_2;$$

but also

$$\tfrac{1}{2}(a12 + a21) = \tfrac{1}{2}(a_{12} - a_{21})(\lambda_1\mu_2 - \lambda_2\mu_1),$$

therefore $$a12 = \overline{a12} + \tfrac{1}{2}s\,.\,(12),$$

where $$-s = (12)\, a12.$$

It is easy to apply this to forms involving more sets of variables, if we remember that in these results the coefficients may themselves be such forms. We have, for example,

$$a123 = \overline{a123} - \tfrac{1}{3}\{(23)\, a\,.\,(23) + (13)\, a\,.\,(13) + (12)\, a\,.\,(12)\},$$

and so, generally,

$$a12\ldots k = \overline{a12\ldots k} - \frac{\Sigma\,(12)a\,.\,(12)}{n} + \frac{\Sigma\,(12)\,(34)a\,.\,(12)\,(34)}{\tfrac{1}{2}n\,(n-1)} - \ldots,$$

the coefficients being the reciprocals of the binomial coefficients*.

Theory of Quadratic Forms.

6. In connection with the quadratic form $a12$, we have already considered the invariant

$$a12 + a21 = s_a\,(12), \text{ where } (12)\, a12 = -s_a.$$

We have thus the formula

$$a21 = -a12 + s_a\,.\,(12) \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

* [In the last two equations of Par. 5, the symbols a are to be considered as abbreviations for $a23$, $a13$, $a12$; $a1234$, &c. Sp.]

To this if we add

$$(13)\, a12 = -a32, \quad (23)\, a12 = -a13 \ldots\ldots\ldots\ldots (2),$$

we shall have exhausted all the invariants and covariants of the first order in the coefficients.

The *discriminant* which is the only invariant of the second order, is given by the square of the quadratic form. We may write

$$a12 \,.\, a12 = -2D_{aa} \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where $$D_{aa} = a_{11}a_{22} - a_{12}a_{21}.$$

We have

$$\begin{aligned} a21 \,.\, a21 &= \{-a12 + s_a \,.\, (12)\}\,\{-a12 + s_a \,.\, (12)\} \\ &= a12 \,.\, a12 - 2s_a\,(12)\, a12 + s_a^2 \,.\, (12)^2 \\ &= a12 \,.\, a12 + 2s_a^2 - 2s_a^2 = a12 \,.\, a12, \end{aligned}$$

as it obviously should; and this may be regarded as a proof of the formula (2).

Moreover,

$$\begin{aligned} a12 \,.\, a21 = a12\,\{-a12 + s_a\,(12)\} &= -a12 \,.\, a12 - s_a^2 \\ &= 2D_{aa} - s_a^2 \ldots\ldots\ldots\ldots\ldots\ldots (4); \end{aligned}$$

and again since

$$\overline{a12} = a12 - \tfrac{1}{2}s \,.\, (12),$$

$$\begin{aligned} \overline{a12} \,.\, \overline{a12} &= a12 \,.\, a12 + s_a^2 + \tfrac{1}{4}s_a^2\,(12) \\ &= a12 \,.\, a12 + \tfrac{1}{2}s_a^2; \end{aligned}$$

therefore $$\overline{D_{aa}} = D_{aa} - \tfrac{1}{4}s_a^2 \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

where $\overline{D_{aa}}$ is the corresponding invariant of the symmetrical function $\bar{a}$.

Since the product of two *even* forms is independent of their order, we have

$$a12 \,.\, a13 = a13 \,.\, a12.$$

But

$$\begin{aligned} a12 \,.\, a13 + a13 \,.\, a12 &= -(23)\, a12 \,.\, a13 \,.\, (23) \\ &= a13 \,.\, 13\,(23) = -2D_{aa}\,(23), \end{aligned}$$

therefore $$a12 \,.\, a13 = -D_{aa}\,(23) \ldots\ldots\ldots\ldots\ldots (6).$$

Hence

$$a12\,.\,a31 = a12\,\{-a13 + s_a\,(13)\} = D_{aa}\,.\,(23) - s_a\,.\,a32 \ldots\ldots (7),$$

$$\begin{aligned} a21\,.\,a31 &= \{-a12 + s_a\,(12)\}\,\{-a13 + s_a\,(13)\} \\ &= -D_{aa}\,(23) + s_a\,(a32 + a23) - s_a^2\,(23) \\ &= -D_{aa}\,(23) = a12\,.\,a13 \ldots\ldots\ldots\ldots\ldots\ldots (8), \end{aligned}$$

$$a21\,.\,a13 = a21\,\{-a31 + s_a\,(13)\} = D_{aa}\,(23) - s_a\,.\,a23 \ldots (9).$$

In the case of a symmetrical function $s = 0$, and the formulæ reduce themselves to the following:

$$a12\,a12 = -2D_{aa},\quad a21 = -a12,\quad a12\,.\,a13 = -D_{aa}\,.\,(23).$$

The following formulæ are also worth noticing:

$$a12\,.\,a34 + a13\,.\,a24 = \{D_{aa}\,.\,(14) - s_a\,.\,a14\}\,(23) \ldots\ldots (10),$$

$$\begin{aligned} a12\,.\,a34 - a21\,.\,a43 &= s_a\,\{a23\,.\,(14) - a14\,.\,(23)\} \\ &= s_a\,\{a41\,.\,(23) - a32\,.\,(14)\} \ldots\ldots (11); \end{aligned}$$

these enable us to make a single or bifid substitution in a product of the form by itself with two different sets of variables.

Passing now to two different forms a, b, we have in the first place the covariant

$$\vartheta_{ab}23 = a12\,.\,b13 = \begin{vmatrix} a_{11}b_{21} - a_{21}b_{11},\ a_{11}b_{22} - a_{21}b_{12} \\ a_{12}b_{21} - a_{22}b_{11},\ a_{12}b_{22} - a_{22}b_{12} \end{vmatrix} (\mu_1\mu_2)(\nu_1\nu_2) \ldots (12),$$

and next the invariant

$$D_{ab} = a_{11}b_{22} - a_{12}b_{21} - a_{21}b_{12} + a_{22}b_{11} = -a12\,.\,b12 \ldots\ldots (13).$$

We may now make the following analysis of the one-place and two-place products of a and b:

$$\left.\begin{aligned} &a12\,.\,b13 = \vartheta 23, \\ &a12\,.\,b31 = a12\,\{-b13 + s_b\,(13)\} = -\vartheta 23 - s_b\,.\,a32, \\ &a21\,.\,b13 = \{-a12 + s_a\,(12)\}\,b13 = -\vartheta 23 - s_a\,.\,b23, \\ &a21\,.\,b31 = \{-a12 + s_a\,(12)\}\,\{-b13 + s_b\,(13)\} \\ &\qquad = \vartheta 23 + s_b\,.\,a32 + s_a\,.\,b23 - s_a s_b\,.\,23 \end{aligned}\right\} (14),$$

$$\left.\begin{aligned} &a12\,.\,b12 = -D_{ab}, \\ &a12\,.\,b21 = a12\,\{-b12 + s_b\,.\,(12)\} = D_{ab} - s_a s_b \end{aligned}\right\} \ldots\ldots (15).$$

Let us now apply to $\mathfrak{Y}$ all those formulæ which we have proved for a single form. Thus from (1), we have

$$\mathfrak{Y}23+\mathfrak{Y}32=-(23)\mathfrak{Y}23(23)=-(23)a12.b13.(23)=a13.b13.(23)$$
$$=D_{ab}(23) \text{ or } s_{\mathfrak{Y}}=D_{ab} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16).$$

Next, from (3),

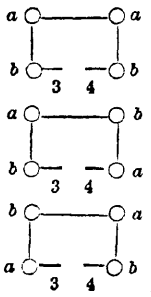
$$-2D_{\mathfrak{Y}\mathfrak{Y}}=\mathfrak{Y}23\,.\,\mathfrak{Y}23=a12\,.\,b13\,.\,a42\,.\,b43=D_{aa}D_{bb}(14)^2$$
$$=-2D_{aa}D_{bb} \text{ or } D_{\mathfrak{Y}\mathfrak{Y}}=D_{aa}D_{bb} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (17).$$

Hence
$$\left.\begin{aligned}\mathfrak{Y}23\,.\,\mathfrak{Y}32&=2D_{aa}D_{bb}-D_{ab}^2,\\ \overline{\mathfrak{Y}23}&=\mathfrak{Y}23-\tfrac{1}{2}D_{ab}(23),\\ \text{and}\qquad \overline{D_{\mathfrak{Y}\mathfrak{Y}}}&=D_{aa}D_{bb}-\tfrac{1}{4}D_{ab}^2,\end{aligned}\right\}\ldots\ldots\ldots\ldots (18),$$

which last agrees with the known formula.

We have next the one-place products of $\mathfrak{Y}$ by itself; namely,

$$\left.\begin{aligned}\mathfrak{Y}23.\mathfrak{Y}24&=-D_{aa}D_{bb}(34)=\mathfrak{Y}32.\mathfrak{Y}42=a12.b13.a52.b54\\ &=a13.b12.a54.b52\\ \mathfrak{Y}23.\mathfrak{Y}42&=+D_{aa}D_{bb}(34)-D_{ab}.\mathfrak{Y}43=a12.b13.a54.b52\\ \mathfrak{Y}32.\mathfrak{Y}24&=+D_{aa}D_{ab}(34)-D_{ab}.\mathfrak{Y}34=a13.b12.a52.b54\end{aligned}\right\}(19);$$



but it is important also to calculate these for the mean value $\overline{\mathfrak{Y}}$; namely, we have

$$\overline{\mathfrak{Y}23}\,.\,\overline{\mathfrak{Y}24}=\{\mathfrak{Y}23-\tfrac{1}{2}D_{ab}(23)\}\{\mathfrak{Y}24-\tfrac{1}{2}D_{ab}(24)\}$$
$$=-D_{aa}D_{bb}(34)+\tfrac{1}{2}D_{ab}(\mathfrak{Y}43+\mathfrak{Y}34)-\tfrac{1}{4}D_{ab}^2(34)$$
$$=-(D_{aa}D_{bb}-\tfrac{1}{4}D_{ab}^2)(34)=-\overline{D_{\mathfrak{Y}\mathfrak{Y}}}\,.\,(34),$$

as it ought to be, confirming equation (18).

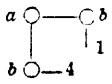
Lastly, from (10) and (11), we get

$$\mathfrak{Y}12\,.\,\mathfrak{Y}34+\mathfrak{Y}13\,.\,\mathfrak{Y}24=\{D_{aa}\,.\,D_{bb}\,.\,(14)-D_{ab}\,.\,\mathfrak{Y}14\}(23)$$
$$\mathfrak{Y}12\,.\,\mathfrak{Y}34-\mathfrak{Y}21\,.\,\mathfrak{Y}43=D_{ab}\{23\,.\,(14)-\mathfrak{Y}14\,(23)\}$$
$$=D_{ab}\{\mathfrak{Y}41\,.\,(23)-\mathfrak{Y}32\,.\,(14)\}\ldots..(20).$$

There are four one-place products of a, a, b; namely,

$$\left.\begin{aligned}
&a12.a32.b34 = D_{aa}.b14, \qquad a12.a32.b43 = D_{aa}.b41,\\
&a12.a23.b34 = -D_{aa}.b14 + s_a.\vartheta 14 + s_a^2.b14 = (s_a^2 - D_{aa})b14 + s_a.\vartheta 14,\\
&a12.a23.b43 = -D_{aa}.b41 - s_a.\vartheta 14 - s_a s_b.a41 - s_a^2 b14 + s_a s_b(14)\\
&\qquad\qquad = (s_a^2 - D_{aa}).b41 - s_a.\vartheta 14 - s_a s_b.a41,
\end{aligned}\right\} (21).$$

When the forms are symmetrical, all these reduce to

$= \pm 2D_{ab}$


$= \pm D_{aa}.b14$

$= \pm \vartheta 12.$

Three forms a, b, c give rise to the new invariant

$$\begin{aligned}
R_{abc} = a12.b23.c31 = &+ a_{11}(b_{21}c_{22} - b_{22}c_{12})\\
&- a_{12}(b_{11}c_{22} - b_{12}c_{12})\\
&- a_{21}(b_{21}c_{21} - b_{22}c_{11})\\
&+ a_{22}(b_{11}c_{21} - b_{12}c_{11}) \ldots\ldots\ldots\ldots (22),
\end{aligned}$$

and the remaining closed products are of the type

$$a21.b23.c31 = \{a12 + s_a(12)\}b23.c31 = -R_{abc} + s_a D_{bc} - s_a s_b s_c.$$

R calculated for the symmetrical functions $\bar{a}$, $\bar{b}$, $\bar{c}$ is

$$\begin{aligned}
\overline{R_{abc}} &= \{a12 - \tfrac{1}{2}s_a(12)\}\{b23 - \tfrac{1}{2}s_b(23)\}\{c31 - \tfrac{1}{2}s_c(31)\}\\
&= R_{abc} + \tfrac{1}{2}(s_a D_{bc} + s_b D_{ca} + s_c D_{ab}) - \tfrac{1}{2}s_a s_b s_c \ldots\ldots\ldots (23).
\end{aligned}$$

It may be mentioned in this place that the determinant made with the coefficients of four forms a, b, c, d has the value

$$\text{Det. } (a, b, c, d) = s_a R_{bcd} - s_b R_{cda} + s_c R_{dab} - s_d R_{abc} = s_a \overline{R_{bcd}} - \&c.,$$

and that no additional invariants are introduced by a greater number of forms.

We now consider the open one-place products of 3 forms. There are evidently four such products having b in the middle; and they are in the first place connected by the formulæ

$$a12.b23.c34 + a42.b23.c31 = -R_{abc}(14),$$

$$a12.b23.c34 + a12.b32.c34 = -s_b a13c34 = s_b \vartheta_{ac}14 + s_a s_b.c14,$$

$$a12.b23.c34 - a34.b23.c12 = -c32\,a34.b21 + c23.a21.b43.$$

[Here the paper abruptly ends. It has been found very difficult to devise a satisfactory way of dealing with the other notes on *graphs* on account of the fragmentary condition in which they were left. Prof. H. J. S. Smith says 'the papers tied up with the fragment on *Quadric forms* are little more than mere jottings. They serve to show that Clifford had applied the method to the Quintic and Sextic. They should be preserved....If the letters by which Clifford denotes certain concomitants are those used in Clebsch's *Binäre Formen*, or in any other accessible work or memoir, it might be possible to rescue the pages relating to the Quintic and Sextic.' By the advice, and at the expense, of Dr Spottiswoode, these fragments have been lithographed, and fifty copies have been printed off for circulation among the principal libraries.]

XXXII.

ON MR SPOTTISWOODE'S CONTACT PROBLEMS*.

THE present communication consists of two parts.

The first part treats of the contact of conics with a given surface at a given point; this class of questions was first treated by Mr Spottiswoode in his paper "On the Contact of Conics with Surfaces," and general formulæ applicable to all such questions were given.

The results of that paper are here reproduced with some additions; with the exception of a few collateral theorems, these are all contained in the following Table:—

†Number of five-point conics through fixed point	= 6
†Order of surface formed by five-point conics through fixed axis	= 8
Number of six-point conics through fixed axis	= 9 ‡
†Number of seven-point conics	= 70

[* From the *Philosophical Transactions* of the Royal Society of London, Vol. CLXIV. Part 2.]

† These results constitute the additions.

‡ [In the Memoir quoted by Professor Clifford, it was stated that the number of conics passing through a given axis and having six-pointic contact with a surface at a given point is ten. In making this statement I overlooked the fact that, in order to put in evidence that a certain quantity was a factor of the equation which determines the positions of the planes of the conics, the equation was multiplied by a quantity D which is a linear function of the position. In reckoning the degree of the equation this factor must of course be discarded. The degree is consequently less by unity than that stated in the Memoir; viz. it is 9, as proved by Professor Clifford.—Sr. July 3, 1873.]